The Markov-Switching Jump Diffusion LIBOR Market Model

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Abstract

In this paper, we introduce an extension to the LIBOR Market model that is suitable to incorporate both sudden market shocks as well as changes in the overall economic climate into the interest rate dynamics. This is achieved by substituting the simple diffusion process of the original LIBOR Market model by a regime-switching jump diffusion. We demonstrate that the new Markov-switching jump diffusion (MSJD) LIBOR Market model can be embedded into a generalized regime-switching Heath-Jarrow-Morton (HJM) model and prove that the considered market is arbitrage-free. We derive pricing formulas for caps, floors, and interest rate swaps using Fourier pricing techniques and show how the model can be calibrated to real data.

Keywords LIBOR Market Model · Jump Diffusion · Markov Switching · Heath-Jarrow-Morton Model · Pricing · Parameter Estimation

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1. Introduction

In 1997, the log-normal LIBOR Market Model (LMM) revolutionized the modeling of interest rates, as the new approach based on the consideration of simple rather than instantaneous forward rates finally allowed for closed-form pricing formulae for the most commonly traded interest rate products. Since its introduction by Brace et al. (1997) and Miltersen et al. (1997), the LMM has experienced an unprecedented raise in popularity and has become the most popular pricing approach among practitioners. Unfortunately, however, it has turned out that the assumption of simple forward rates following log-normal dynamics is not sufficient to account for complicated market movements or non-flat implied volatility surfaces (see, Rebonato (2002)). As a consequence, a large amount of extensions has been brought forth over the course of years and, among others, the simple log-normal processes of the original model were replaced by displaced diffusions (see Joshi and Rebonato (2003)), Lévy processes (see Eberlein and Özkan (2005)), generalized jump diffusions (see Glasserman and Kou (2003) and Belomestny and Schoenmakers (2011)), Markov-switching geometric Brownian motions (see Elliott and Valchev (2004)), processes with stochastic volatility (see Andersen and Brotherton-Ratcliffe (2005) and Belomestny et al. (2010)) and general semimartingales (see Jamshidian (1999)). Even extensions accounting for default risk were introduced (see Eberlein et al. (2006) and Eberlein and Grbac (2011)).

The purpose of this paper is to introduce an extension to the log-normal LMM that successfully merges two of the most promising concepts: generalized jump diffusions and Markov-switching processes. In the following, the approach shall be referred to as the Markov-Switching Jump Diffusion (MSJD) extension to the LMM. Through this model, it is possible to both incorporate sudden market shocks as well as changes in the overall economic climate, or structural breaks, into the interest rate dynamics. On the one hand it has been demonstrated by Belomestny and Schoenmakers (2011) that modeling simple interest rates through jump diffusions is not only suited to reflect sudden jumps observed in the market dynamics, but also allows to successfully capture the non-flat implied volatility surfaces typically observed in the interest rate derivatives markets. On the other hand, Rebonato and Joshi (2002) and Rebonato (2003) present considerable evidence indicating that there are in fact different economic phases. This observation is incorporated into the model by the Markov-switching feature: All jump diffusion parameters are assumed to be dependent on an underlying finite-state space Markov chain moving according to the overall economic development.

By exploiting the relation between bond prices, forward rates, and simple rates, we demonstrate that such an extension to the original model can be embedded into a generalized Markov-switching Heath-Jarrow-Morton model and hereby prove that the considered market is arbitrage-free. We furthermore show how interest rate swaps and their derivatives can be related to the model. With measure changes playing a central role in the derivation of the model, we investigate the consequences of these changes on all modeled entities as well as the underlying Markov chain. Despite the apparent complexity of our approach, we demonstrate that the pricing and calibration within a Markov-switching jump diffusion model for interest rates is nonetheless possible and yields satisfactory results. Using the Fourier pricing technique, we derive pricing formula and calibrate the model to real market data.[1]

This paper is divided into six sections. Section 2 is meant to recall the most relevant concepts when working with the LIBOR market model – jump diffusion processes, Girsanov's Theorem and the Change-of-Numéraire Technique. Next, Section 3 gives a quick introduction to the log-normal LIBOR market model of Brace et al. (1997) and Miltersen et al. (1997) as well as the swap market model of Jamshidian (1997). This is followed by Section 4, where we rigorously derive an arbitrage-free framework for the MSJD extension to the original model. It is shown how the dynamics of different LIBOR rates can be interrelated and the special role of the Markov chain under measure changes is investigated. Section 5 demonstrates how the modeling of swap dynamics can be embedded into the MSJD extension of the LMM. Then, Section 6 investigates how some of the most important categories of interest rate derivatives, caps/floors and swaptions, can be priced within the MSJD framework. Last, but not least, Section 7 explains how the interest rate dynamics of the proposed extension can be successfully calibrated to market data.

[1] The modeling of LIBOR rates within a Markov-switching jump diffusion framework is also considered in Steinrücke et al. (2013). The main difference between the considerations presented there and this paper is that the former gives an intuitive introduction to Markov-switching jump diffusions, whereas this paper presents the mathematically rigorous framework behind the intuition. Also, swap rates and swaption pricing are not considered in Steinrücke et al. (2013), and calibration issues are only touched upon.

2. Mathematical Preliminaries

Let the finite time horizon $T^* > 0$ be given. The interest rate market is modeled on the complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F}$ on $\mathcal{F}$ satisfies the usual conditions of right-continuity and completeness, and $\mathbb{P}$ denotes the physical measure of the market. It is furthermore assumed that the market is frictionless with bank account $(B_t)_{t\in[0,T^*]}$, $B_0 = 1$, and zero-coupon bonds $(B(t,T))_{t\in[0,T]}$ trading for every maturity $0 \leq T \leq T^*$. Also, the ad-hoc assumption is made that all processes involved are specified in such a way that all operations to be performed (differentiation in the T-variable, differentiation in the t-variable under the integral sign and interchange of order of integration) are well-defined. Note that due to the assumption of a finite time horizon, any local martingale can be treated as though being a martingale (see Björk et al. (1997)).

2.1. Jump Diffusion Processes

Stochastic calculus for jump diffusion processes is essential to the development of the upcoming Markov-switching jump diffusion (MSJD) extension to the LIBOR market model. This subsection is intended to give a short introduction to the most important notions needed in this context. To this end, let $\mathcal{H}$ be an arbitrary probability measure equivalent to $\mathbb{P}$, both defined on $\left(\mathbb{R}^k, \mathcal{B}\left(\mathbb{R}^k\right)\right)$. Furthermore, let μ be an integer-valued random measure on the mark space $\left([0,T^*] \times \mathbb{R}^k, \mathscr{B}([0,T^*]) \otimes \mathcal{B}\left(\mathbb{R}^k\right)\right)$ and $\nu^{\mathcal{H}}$ its compensator measure with respect to $\mathcal{H}$. For all random functions

$$\gamma \in \Big\{g : \Omega \times [0,T^*] \times \mathbb{R}^k \to \mathbb{R}^n; g \text{ predictable },$$
$$\int_0^t \int_{\mathbb{R}^k} \min\left(g^2(s,z), |g(s,z)|\right) \nu^{\mathcal{H}}(ds, dz) < \infty \text{ a.s.}\Big\},$$

the stochastic integral

$$\int_0^t \int_{\mathbb{R}^k} \gamma(s,z) \left(\mu - \nu^{\mathcal{H}}\right)(ds, dz)$$

defines a unique, purely *discontinuous* (local) martingale (see, e.g., Jacod and Shiryaev (2002)). For $W^{\mathcal{H}}$ a d-dimensional standard Brownian motion under $\mathcal{H}$ and $\delta : \Omega \times [0,T^*] \to \mathbb{R}^{n\times d}$ a random process satisfying $\int_0^t \|\delta(s)\|^2 ds < \infty$, the stochastic integral $\int_0^t \delta(s)\, dW^{\mathcal{H}}(s)$ is a *continuous* (local) martingale (see, e.g., Klenke (2008), Theorem 25.18). If additionally, $\alpha : \Omega \times [0, T^*] \to \mathbb{R}^n$ is a predictable process satisfying $\int_0^t \|\alpha(s)\| \, ds < \infty$, then the process

$$Y(t) = \int_0^t \alpha(s) \, ds + \int_0^t \delta(s)' \, dW^{\mathcal{H}}(s) + \int_0^t \int_{\mathbb{R}^k} \gamma(s,z) \left(\mu - \nu^{\mathcal{H}}\right)(ds, dz) \tag{2.1}$$

defines a special semimartingale which is almost surely finite for all $t \in [0, T^*]$. The adaption of Itō's Lemma for semimartingales of type (2.1) reads as follows (compare, e.g., Øksendal and Sulem (2007)):

Theorem 2.1 (Itō's Lemma for Special Semimartingales of Type (2.1))**.**
Let f be a $\mathcal{C}^{1,2}\big([0, T^*] \times \mathbb{R}^n, \mathbb{R}\big)$-function and $Y = (Y_1, \ldots, Y_n)$ be an n-dimensional semimartingale given as in (2.1)*. Then $(f(t, Y(t)))_{t \in [0, T^*]}$ is a semimartingale as well, and satisfies*

$$\begin{aligned} df(t, Y(t)) = & \frac{\partial f(t, Y(t))}{\partial t} dt + \sum_{i=1}^n \frac{\partial f(t, Y(t))}{\partial Y_i} \left(\alpha_i(t) \, dt + \delta_i(t)' \, dW^{\mathcal{H}}(t)\right) \\ & + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f(t, Y(t))}{\partial Y_i \partial Y_j} \left(\delta(t) \, \delta(t)'\right)_{i,j} dt \\ & + \int_{\mathbb{R}^k} \left[f(t, Y(t-) + \gamma(t,z)) - f(t, Y(t-))\right] \left(\mu - \nu^{\mathcal{H}}\right)(dt, dz) \\ & + \int_{\mathbb{R}^k} \Big[f(t, Y(t-) + \gamma(t,z)) - f(t, Y(t-)) \\ & \qquad - \sum_{i=1}^n \gamma_i(t,z) \frac{\partial f(t, Y(t-))}{\partial Y_i}\Big] \nu^{\mathcal{H}}(dt, dz) \, . \end{aligned}$$

An important application of Itō's Lemma lies in the context of stochastic exponentials: For a given one-dimensional semimartingale Y, one would like to find a càdlàg adapted process Z which solves the stochastic differential equation (SDE)

$$dZ(t) = Z(t-) \, dY(t) \, , \qquad Z(0) = z_0. \tag{2.2}$$

In the case, where Y is given as in (2.1), (2.2) reads

$$dZ(t) = Z(t-) \left[\alpha(t) \, dt + \delta(t)' \, dW^{\mathcal{H}}(t) + \int_{\mathbb{R}^k} \gamma(t,z) \left(\mu - \nu^{\mathcal{H}}\right)(dt, dz)\right], \quad Z(0) = z_0. \tag{2.3}$$

The solution to the general problem (2.2) is called the *stochastic exponential* or *Doléans-Dade exponen-*

tial and is usually denoted by $\mathcal{E}(Y) = (\mathcal{E}_t(Y))_{t\geq 0}$ (see, e.g., Theorem 13.5 in Elliott (1982)). In the special case (2.3), it is most frequently referred to as a *jump diffusion*. It can be shown that the solution exists and is unique (following V, Theorem 6 Protter (2005)). Under application of Itō's Lemma 2.1, one may furthermore derive the solution to (2.3) to be given as

$$
\begin{aligned}
Z(t) = z_0 \cdot \exp\Big(& \int_0^t \big(\alpha(s) - \frac{1}{2}\left\|\delta(s)\right\|^2\big)ds + \int_0^t \delta(s)' \, dW^{\mathcal{H}}(s) \\
& + \int_0^t \int_{\mathbb{R}^k} \ln\left(1+\gamma(s,z)\right)\left(\mu - \nu^{\mathcal{H}}\right)(ds,dz) \\
& + \int_0^t \int_{\mathbb{R}^k} \big[\ln\left(1+\gamma(s,z)\right) - \gamma(s,z)\big]\nu^{\mathcal{H}}(ds,dz)\Big),
\end{aligned}
\tag{2.4}
$$

As (2.3) may be rewritten as

$$
\begin{aligned}
dZ(t) = & Z(t-)\,\alpha(t)\,dt + Z(t-)\,\delta(t)\,dW^{\mathcal{H}}(t) \\
& + \int_{\mathbb{R}^k} Z(t-)\,\gamma(t,z)\left(\mu - \nu^{\mathcal{H}}\right)(dt,dz), \quad Z(0) = z_0,
\end{aligned}
\tag{2.5}
$$

with $Z(t-)\,\alpha(t)$, $Z(t-)\,\delta(t)$ and $Z(t-)\,\gamma(t,z)$ again predictable, integrable processes, (2.5) may be read as a special semimartingale of type (2.1). Consequently, Itō's Lemma 2.1 may be correspondingly applied by replacing the coefficient functions in (2.1) by $Z(t-)\,\alpha(t)$, $Z(t-)\,\delta(t)$ and $Z(t-)\,\gamma(t,z)$.

2.2. The Change-of-Numéraire Technique and Girsanov's Theorem

Pricing in the interest rate market involves changes to measures associated with numéraires different than the simple bank account B. For this, the *Change-of-Numéraire Technique* is needed (see, e.g., Brigo and Mercurio (2006) or Zagst (2001)):

Theorem 2.2 (Change-of-Numéraire Technique)**.**
Let $(Y(t))_{t\in[0,T^*]}$ be a primary traded asset of the market and $\mathcal{Q}$ an equivalent martingale measure (EMM) under which $\left(B_t^{-1}Y(t)\right)_{t\in[0,T^*]}$ follows a martingale. Let $A = (A(t))_{t\in[0,T^*]}$ and $E = (E(t))_{t\in[0,T^*]}$ be two arbitrary numéraires, satisfying that the discounted processes $\left(B_t^{-1}A(t)\right)_{t\in[0,T^*]}$ and $\left(B_t^{-1}E(t)\right)_{t\in[0,T^*]}$ are both $\mathcal{Q}$-martingales. Then, the following holds:

- *There exists an equivalent probability measure $\mathcal{Q}^A$,*

$$\left.\frac{d\mathcal{Q}^A}{d\mathcal{Q}}\right|_{\mathcal{F}_t} := \frac{A(t)}{A(0)\,B_t}, \qquad t \in [0, T^*],$$

such that $\left(A(t)^{-1}Y(t)\right)_{t\in[0,T^*]}$ is a $\mathcal{Q}^A$-martingale.

- *The Radon-Nikodým derivative of $\mathcal{Q}^E$ with respect to $\mathcal{Q}^A$ is given as*

$$\left.\frac{d\mathcal{Q}^E}{d\mathcal{Q}^A}\right|_{\mathcal{F}_t} := \frac{E(t)}{A(t)} \cdot \frac{A(0)}{E(0)}, \qquad \forall\, t \in [0, T^*].$$

- *For any contingent claim $D = D(T)$ with underlying Y, the time-t-price is given as*

$$B_t \mathbb{E}_{\mathcal{Q}}\left[\frac{D}{B_T}\middle|\mathcal{F}_t\right] = A(t)\,\mathbb{E}_{\mathcal{Q}^A}\left[\frac{D}{A(T)}\middle|\mathcal{F}_t\right] = E(t)\,\mathbb{E}_{\mathcal{Q}^E}\left[\frac{D}{E(T)}\middle|\mathcal{F}_t\right].$$

The numéraires used in the context of the LMM are (zero-coupon) bonds $(B(t,T))_{t\in[0,T]}$ with maturities $0 < T \leq T^*$, for which it is natural to assume that $B(t,T) > 0$. The measure $\mathcal{Q}^T$ associated with the numéraire $(B(t,T))_{t\in[0,T]}$ is called the *T-forward measure*. As all relevant processes in the upcoming model will be jump diffusions of the type (2.3), the following version of Girsanov's theorem comes in handy (see Schönbucher (2003)). Note that measure changes have no impact on the jump measure, but only on the respective compensator $\nu^{\mathcal{H}}$:

Theorem 2.3 (Girsanov's Theorem)**.**
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{H})$ be a complete stochastic basis with $\mathcal{H}$ an arbitrary probability measure. Furthermore, let $W^{\mathcal{H}}(t)$ be a d-dimensional $\mathcal{H}$-Brownian motion and μ an integer-valued random measure with mark space $([0,T^*] \times E, \mathscr{B}([0,T^*]) \otimes \mathcal{E})$ and $\mathcal{H}$-compensator $\nu^{\mathcal{H}}(dt,dz) = \lambda^{\mathcal{H}}(t)\,k^{\mathcal{H}}(t,dz)\,dt$. $\lambda^{\mathcal{H}}$ and $k^{\mathcal{H}}$ denote the predictable jump intensity and the marker distribution, respectively, and it is assumed that $k^{\mathcal{H}}(t,A)$ is predictable $\forall\, A \in \mathcal{E}$. Also, let θ be a d-dimensional predictable process and $\Phi(t,z)$ a nonnegative predictable function satisfying the usual integrability assumptions. Define the process $Z(t)$ by $Z(0) = 1$ and

$$\frac{dZ(t)}{Z(t-)} = \theta(t)\,dW^{\mathcal{H}}(t) + \int_E (\Phi(t,z) - 1)\left(\mu(dz,dt) - \nu^{\mathcal{H}}(dz,dt)\right).$$

It is assumed that $\mathbb{E}^{\mathcal{H}}[Z(t)] = 1$ *for all* $0 \leq t \leq T$. *Define the equivalent measure* $\mathcal{R} \sim \mathcal{H}$ *by the Radon-Nikodým derivative* $d\mathcal{R}/d\mathcal{H}|_{\mathcal{F}_t} = Z(t)$. *Then, the following holds:*

(i) The process $W^{\mathcal{R}}$ *is a* $\mathcal{R}$*-Brownian motion,*

$$dW^{\mathcal{R}}(t) = dW^{\mathcal{H}}(t) - \theta(t)\,dt.$$

(ii) The a.s. unique predictable $\mathcal{R}$*-compensator of* μ *is given as*

$$\nu^{\mathcal{R}}(dz, dt) := \Phi(t, z)\,\nu^{\mathcal{H}}(dz, dt)\,.$$

The corresponding jump intensity and the marker distribution are

$$\lambda^{\mathcal{R}}(t) = \phi(t)\,\lambda^{\mathcal{H}}(t) \quad \textit{and} \quad k^{\mathcal{R}}(t, dz) = Z_E(z)\,k^{\mathcal{H}}(t, dz)\,,$$

respectively, where $\phi(t) := \int_E \Phi(t, z)\,k^{\mathcal{H}}(t, dz)$, *and* $Z_E(z) := \Phi(t, z)/\phi(t)$ *for* $\phi(t) > 0$, $Z_E(z) = 1$ *otherwise.*

3. The Log-Normal LIBOR Market Model and the Embedding of Swaption Pricing

While there has been a wide range of extensions to the instantaneous forward rate model of Heath et al. (1992), the approach bears the intrinsic problem of dealing with an infinite number of interest rates that are not directly observable on the interest market. Even more severe, the most basic interest rate derivatives (caps and swaptions) cannot be evaluated via a closed-form pricing formula. It was mainly these problems that gave rise to a new modeling approach – the LIBOR market model as introduced by Brace et al. (1997) and Miltersen et al. (1997). In contrast to before, the authors proposed to concentrate on simple instead of instantaneous forward rates. Jamshidian (1997) soon after applied the underlying idea to modeling in the swap market. The following two subsections are meant to give a short overview on their models. More detailed introductions to the topic can be found in Brigo and Mercurio (2006), Filipovic (2009), Rebonato (2002) and Zagst (2001).

3.1. The Log-Normal LIBOR Market Model (LMM)

Let $0 = T_0 < T_1 < \ldots < T_N$ be a fixed tenor structure, with constant tenor $\delta \equiv T_{i+1} - T_i$, $i = 1, \ldots, N-1$. The *forward LIBOR rate* or *simple forward rate* $L_i(t) := L(t, T_i, T_{i+1})$ with *maturity* T_i and *expiry* T_{i+1} is the simple interest rate that an investor can lock in at time t for the future time interval $[T_i, T_{i+1}]$, and is given by the relation

$$1 + \delta \cdot L_i(t) = \frac{B(t, T_i)}{B(t, T_{i+1})}. \tag{3.1}$$

This is indeed the simple forward rate, as one can easily construct a portfolio that allows for replication (see, e.g., Shreve (2008), p. 423, 424). Taking the expiry of one LIBOR rate as the maturity of the next yields an array of $N-1$ LIBOR rates, $L_1, \ldots, L_{N-1}$.[2]

In the most basic setting of the LMM, as it was considered in the seminal paper of Brace et al. (1997), it is assumed that the only source of randomness in the market is a d-dimensional standard Brownian motion. The market filtration $\mathbb{F}$ is the augmented and completed version of the filtration generated by it. Observing that the LIBOR market model can naturally be embedded into the HJM model, Brace et al. (1997) showed that the model can be assumed to be free of arbitrage and that there exists a spot martingale measure $\mathcal{Q}$, under which all bonds discounted with the money market account B are martingales. Following from this, each LIBOR rate L_i can be demonstrated to follow a martingale under the forward measure $\mathcal{Q}^{i+1}$ associated with the bond price $(B(t, T_{i+1}))_{t \in [0,T]}$ acting as numéraire. As a result, the martingale representation theorem for Brownian markets (see, e.g., Zagst (2001), p. 31) implies that each forward LIBOR rate L_i can be modeled as a geometric Brownian motion with drift 0 under the respective forward measure $\mathcal{Q}^{i+1}$,

$$dL_i(t) = L_i(t) \cdot \sigma_i(t)' \, dW^{i+1}(t), \qquad L_i(0) = l_i, \tag{3.2}$$

where W^{i+1} is a d-dimensional $\mathcal{Q}^{i+1}$-Brownian motion, σ_i a predictable d-dimensional vector function satisfying $\int_0^{T_i} \|\sigma_i(s)\|^2 \, ds < \infty$ $\mathcal{Q}^{i+1}$-a.s. and l_i is determined according to (3.1) evaluated at 0. As in

[2] Most authors substitute the accurate term "forward LIBOR rate" for $L_i(t) = L(t, T_i, T_{i+1})$ by the more convenient shortened expression "LIBOR rate". Strictly speaking, this is only appropriate when $t = T_i$, but since no great confusion should be expected, we will also follow this convention.

(2.4), the solution to the SDE (3.2) is then given by the stochastic exponential

$$L_i(t) = l_i \cdot \exp\left(\int_0^t \sigma_i(s)' \, dW^{i+1}(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 \, ds\right).$$

Given the exponential form, the i-th LIBOR rate is non-negative, whenever $L_i(0) \geq 0$. This can be ensured by an initial term structure of the zero-coupon bonds $B(0, T_i)$, $i = 1, \ldots, N$ which is positive and non-increasing in maturity, $0 < B(0, T_N) \leq \ldots \leq B(0, T_1)$.

3.2. The Log-Normal Swap Market Model (LSM)

The swap market model naturally builds upon the LMM. For a set of pre-specified, successive dates $T_\alpha, T_{\alpha+1}, \ldots, T_\beta \in \{T_1, \ldots, T_N\}$, $1 \leq \alpha < \beta \leq N$, a *payer interest rate swap (PIRS)* is a contract that involves receiving floating for fixed-leg payments: Let the notional of the contract equal 1. With K being a fixed annualized interest rate, the owner of the contract makes a payment δK at every instant $T_{i+1} \in \{T_{\alpha+1}, \ldots, T_\beta\}$, while receiving a corresponding floating payment $\delta L_i(T_i)$ at each of these time points. In contrast, a *receiver interest rate swap (RIRS)* entails receiving fixed for floating payment. The interest rates of the floating leg reset at dates $T_\alpha, T_{\alpha+1}, \ldots, T_{\beta-1}$, while they are being paid at $T_{\alpha+1}, \ldots, T_\beta$, respectively. By risk-neutral valuation, the change of numéraire technique 2.2 and the martingale property of L_i, it is easy to see that the value PFS_t of a PIRS contract at time t is

$$\begin{aligned}\mathrm{PFS}_t &= \sum_{i=\alpha}^{\beta-1} B(t, T_{i+1})\, \delta\Big(\mathbb{E}_{Q^{i+1}}\left[L_i(T_i) \,\middle|\, \mathcal{F}_t\right] - K\Big) = \sum_{i=\alpha}^{\beta-1} B(t, T_{i+1})\, \delta\left[L_i(t) - K\right] \\ &= \sum_{i=\alpha}^{\beta-1} \left(B(t, T_i) - (1 + \delta K)\, B(t, T_{i+1})\right),\end{aligned}$$

where definition (3.1) was used to rewrite the expression in the second line. The price of a RIRS can be determined accordingly. The *forward swap rate* $S_{\alpha,\beta}(t)$ is then defined as the fixed rate that makes the value of the swap at time t equal to zero,

$$S_{\alpha,\beta}(t) := \frac{\sum_{i=\alpha}^{\beta-1} B(t, T_i) - B(t, T_{i+1})}{\delta \sum_{i=\alpha}^{\beta-1} B(t, T_{i+1})} = \frac{B(t, T_\alpha) - B(t, T_\beta)}{\delta \sum_{i=\alpha}^{\beta-1} B(t, T_{i+1})}. \tag{3.3}$$

Using a similar argument to the one used to derive the martingale property of LIBOR rates under their corresponding forward measures, Jamshidian (1997) demonstrated that $S_{\alpha,\beta}$ is a martingale with respect to the so-called *swap measure* $\mathcal{Q}^{\alpha,\beta}$ associated with the numéraire

$$C_{\alpha,\beta}(t) = \delta \sum_{i=\alpha}^{\beta-1} B(t, T_{i+1}), \tag{3.4}$$

the so-called *annuity*. By same reasoning as in the LIBOR case, swap rates may hence be assumed to follow dynamics

$$dS_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)\, \sigma_{\alpha,\beta}(t)\, dW^{\alpha,\beta}(t), \tag{3.5}$$

with $W^{\alpha,\beta}$ denoting a $\mathcal{Q}^{\alpha,\beta}$ Wiener process and $\sigma_{\alpha,\beta}$ a strictly positive, deterministic function satisfying $\int_0^{T_\alpha} \|\sigma_{\alpha,\beta}(s)\|^2\, ds < \infty$ $\mathcal{Q}^{\alpha,\beta}$-a.s. The initial value of $S_{\alpha,\beta}$ is given through (3.3) evaluated at 0.

3.3. Pricing of Caps/Floors and Swaptions

An important implication of the development of the market model approach was that it finally allowed for the closed-form evaluation of caps/floors and swaptions in terms of a Black (1976)-type formula. Recall that an *interest rate caplet (floorlet)* is an instrument that protects its owner against too high (low) interest rates. For notional 1, the owner of a caplet (floorlet) on the i-th LIBOR rate L_i receives a payment at time T_{i+1} that equals the amount in which $\delta \cdot L_i(T_i)$ exceeds (falls below) the pre-specified strike δK, such that the payoff of the contract equals

$$\delta \cdot (L_i(T_i) - K)^+ \qquad \left[\, \delta \cdot (K - L_i(T_i))^+ \,\right].$$

Since a change in sign is the only difference between the payoff of the instruments "caplet" and "floorlet", it suffices to concentrate on the former. By the usual principle of risk-neutral valuation and the Change-of-Numéraire Technique 2.2 it can be easily shown that the time-t price of a caplet on the i-th LIBOR rate with strike K is given by

$$\text{Caplet}_i(t) = \delta \cdot B(t, T_{i+1}) \cdot \mathbb{E}_{\mathcal{Q}^{i+1}}\left[(L_i(T_i) - K)^+ \middle| \mathcal{F}_t \right]$$

Under specification (3.2), a Black-Scholes-type argument (see Schoenmakers (2005)) yields

$$\text{Caplet}_i(t) = \delta \cdot B(t, T_{i+1}) \cdot \left[L_i(t) \cdot \mathcal{N}\big(\tilde{d}_1\big) - K \cdot \mathcal{N}\big(\tilde{d}_2\big) \right], \tag{3.6}$$

where $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution and

$$\widetilde{d}_{1,2} = \frac{\ln\left(\frac{L_i(t)}{K}\right) \pm \frac{1}{2}\int_t^{T_i} \|\sigma_i(s)\|^2 \, ds}{\sqrt{\int_t^{T_i} \|\sigma_i(s)\|^2 \, ds}}. \tag{3.7}$$

An *interest rate cap (floor)* is a strip of caplets over a set of time periods $[T_i, T_{i+1}]$, $T_0 < T_1 \ldots < T_N$, where at the end of each period the buyer of the contract receives a payment, whenever the interest rate fixed at the beginning of each period exceeds (falls below) the pre-specified strike price K. A cap is basically a series of caplets with the same underlying strike K, and hence, the time-t price equals

$$\text{Cap}(t) = \sum_{i=1,\ldots,n-1; T_i \geq t} \text{Caplet}_i(t).$$

The second main class of derivatives in interest rate markets are *swaptions*, where these instrument are – as the name indicates – options on underlying interest rate swaps: A *European payer (receiver) swaption* with strike K is a contract that gives the buyer the right to enter into a payer (receiver) swap with fixed rate K at the future time point T_α. The first reset time is T_α, and the payments start at $T_{\alpha+1}$, going until T_β. In a similar argument to the pricing of caplets, the time-t-price of a payer forward swaption is given as

$$\text{PSwaption}_{\alpha,\beta}(t) = C_{\alpha,\beta}(t) \left[S_{\alpha,\beta}(t) \cdot \mathcal{N}\big(\hat{d}_1\big) - K\mathcal{N}\big(\hat{d}_2\big) \right], \tag{3.8}$$

where $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution and

$$\hat{d}_{1,2} = \frac{\ln\left(\frac{S_{\alpha,\beta}(t)}{K}\right) \pm \frac{1}{2}\int_t^{T_\alpha} \|\sigma_{\alpha,\beta}(s)\|^2 \, ds}{\sqrt{\int_t^{T_\alpha} \|\sigma_{\alpha,\beta}(s)\|^2 \, ds}}.$$

Note at this point that the market models LMM and LSM are not compatible with each other. This means that while LIBOR rates follow martingales with respect to their corresponding forward measure, swap rates do not, and vice versa (see, e.g., Brigo and Mercurio (2006), Section 6.8). As a consequence, practitioners oftentimes decide to model the LIBOR rates according to (3.2) and price caplets (and floorlets) according to the analytical evaluation formula (3.6), whereas swap rates are suitably approximated and then used to derive approximate swaption prices (compare also Section 5).

4. The Markov-Switching Jump Diffusion (MSJD) Extension of the LMM

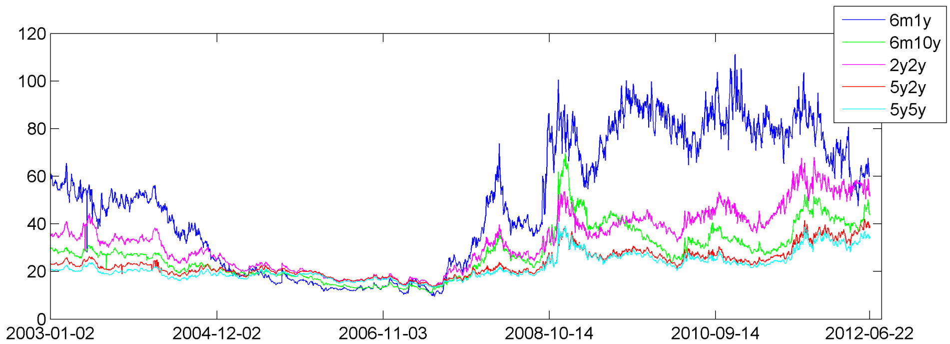

Figure 1: ATM implied volatilities (in %) for 6m1y, 6m10y, 2y2y, 5y2y and 5y5y swaptions with quarterly occurring reset dates.

When examining the evolvement of cap and swaption prices over time, there are several indicators that simple log-normal dynamics for LIBOR and swap rates might not be suited to adequately reproduce the evolvement of LIBOR rates over time. Figure 1 depicts, e.g., the evolvement of the implied volatilities of four different swaptions with start and maturity dates between 6 months and 10 years in the time interval 2003/01/01-2012/06/22. Even to the mere eye, it appears obvious that swaption prices are more volatile during certain times than they are during others. Furthermore, a log-normal model as in (3.5) seems to only offer very limiting explanatory power, when it comes to the jumps observed in Figure 1, as large displacement are very unlikely to occur under normally distributed increments. Very similar observations can be made for time series of caplet volatilities. These apparent shortcomings in the log-normal market models motivate the following proposed extension to the log-normal LMM, where the diffusion processes in (3.2) are substituted by so-called *Markov-switching jump diffusions*. As the name indicates, these processes are jump diffusions whose coefficient functions as well as compensator measures are driven by the movement of some underlying finite-state continuous Markov chain representing the overall market movement. We shall call this model the Markov-switching jump diffusion (MSJD) extension to the log-normal LIBOR market model. Observe in the following that, in the special case that the jump part equals 0 and the Markov chain only takes on one state, the MSJD extension coincides with the original log-normal LMM.

4.1. Presenting the Extended Framework

Let the bond structure be positive and non-increasing in the maturity. As before, we assume the existence of a bank account $(B_t)_{t\in[0,T^*]}$ and a risk-neutral measure $\mathcal{Q}$ with respect to which all discounted bonds $(B(t,T)/B_t)_{t\in[0,T]}$, $0 < T \leq T^*$, follow martingales. In extension to the original log-normal LMM, let X be a continuous, time-homogeneous, finite Markov chain, taking values in the standard basis $E = \{e_1, \dots, e_M\}$ of the Euclidean space $\mathbb{R}^M$. The infinitesimal generator of X with respect to the terminal measure $\mathcal{Q}^N$ associated with $(B(t,T_N))_{t\in[0,T_N]}$ is denoted by $\mathcal{A}$, $\mathbb{F}^X$ is the filtration generated by X. It can be shown that X has the semimartingale representation,

$$X_t = X_0 + \int_0^t \mathcal{A}' X_s ds + M_t \tag{4.1}$$

where $M = (M_t)_{t\geq 0}$ is a right-continuous, square-integrable $\mathbb{R}^M$-valued martingale with respect to $\left(\mathbb{P}, \mathbb{F}^X\right)$ (compare, e.g., Elliott et al. (1994)). Also, let μ be a random jump measure defined on the mark space $[0,T^*]\times\mathbb{R}^k$, which is taken to be of finite activity i.e. $\mu\left([0,t]\times\mathbb{R}^k\right) < \infty$ for all $t \in [0,T^*]$.

In extension to (3.1), we propose to model each LIBOR rate L_i, $i = 1, \dots, N-1$, as a Markov-switching jump diffusion, such that every L_i is governed by the SDE

$$\frac{dL_i(t)}{L_i(t-)} = \sigma_i(t, X_{t-})' \, dW^{i+1}(t) + \int_{\mathbb{R}^k} \psi_i(t, X_{t-}, z) \left(\mu - \nu^{i+1}_{X_{t-}}\right)(dt, dz), \tag{4.2}$$

with W^{i+1} a d-dimensional Brownian motion and $\nu^{i+1}_{X_{t-}}$ the predictable $\mathcal{Q}^{i+1}$-compensator of μ. σ_i denotes the regime-dependent volatility and ψ_i the regime-dependent jump function associated with the jump term. The objects involved (i.e., processes, measures and compensators) are to satisfy the subsequent assumptions:

(I) X is the only source of randomness for the volatilities and jump functions. For all $i \in$

$\{1, \ldots, N-1\}$, these are defined as

$$\sigma_i(t) = \sigma_i(t, X_{t-}) = \sum_{j=1}^{M} \langle X_{t-}, e_j \rangle \, \sigma_i(t, e_j), \qquad t \in [0, T_i],$$
$$\psi_i(t, z) = \psi_i(t, X_{t-}, z) = \sum_{j=1}^{M} \langle X_{t-}, e_j \rangle \, \psi_i(t, e_j, z), \qquad t \in [0, T_i]$$

with $\langle ., . \rangle$ denoting the usual scalar product, and volatilities and jump functions satisfying $\sum_{j=1}^{M} \int_0^{T_i} \|\sigma_i(s, e_j)\|^2 \, ds < \infty$, for $i = 1, \ldots, N-1$ and $\sum_{j=1}^{M} \int_0^{T_i} \int_{\mathbb{R}^k} |\psi_i(s, j, z)| \, \nu_j^{i+1}(ds, dz) < \infty$, for $i = 1, \ldots, N-1$. For $t > T_i$, we set $\sigma_i(t, X_{t-}) \equiv 0$ and $\psi_i(t, X_{t-}, z) \equiv 0$, for all $i = 1, \ldots, N-1$.

(II) Conditional on the Markov chain X, the $\mathcal{Q}^N$-Wiener process W^N and the $\mathcal{Q}^N$-compensated jump measure $\mu - \nu_{X_{t-}}^N$ are independent.

(III) The $\mathcal{Q}^N$-compensator $\nu^N(dt, dz)$ of μ is the predictable compensator associated with a homogeneous Markov-switching marked Poisson process,

$$\nu_{X_{t-}}^N(dt, dz) = k^N(X_{t-}, dz) \, \lambda^N(X_{t-}) \, dt = \sum_{j=1}^{M} \langle X_{t-}, e_j \rangle k^N(e_j, dz) \, \lambda^N(e_j) \, dt$$

with $\lambda^N(e_j)$ being the jump intensity and $k^N(e_j, dz)$ the conditional distribution of the markers in state e_j.

The rationale behind requirements (II) and (III) will become evident in Subsection 4.3. For requirement (I), observe that volatilities, jump functions and compensators are dependent on X_{t-} rather than on X_t which ensures the predictability of the coefficient function. In combination with the requirement that μ is integer-valued, (4.2) hence defines a special semimartingale of the type (2.3). The solution to the stochastic differential equation in (4.2) is accordingly given by the Doléans-Dade exponential

$$\begin{aligned} L_i(t) = L_i(0) \cdot \exp\Big(&-\frac{1}{2} \int_0^t \|\sigma_i(s, X_{s-})\|^2 \, ds + \int_0^t \sigma_i(s, X_{s-})' dW^{i+1}(s) \\ &+ \int_0^t \int_{\mathbb{R}^k} \ln(1 + \psi_i(s, X_{s-}, z)) \left(\mu - \nu_{X_{s-}}^{i+1}\right)(ds, dz) \\ &+ \int_0^t \int_{\mathbb{R}^k} \left[\ln(1 + \psi_i(s, X_{s-}, z)) - \psi_i(s, X_{s-}, z)\right] \nu_{X_{s-}}^{i+1}(ds, dz) \Big). \end{aligned}$$

It is worthwhile noting that, similar to the log-normal model, the exponential form ensures non-negativity for each LIBOR rate L_i, whenever $L_i(0) \geq 0$, $1 \leq i \leq N-1$. As we postulated that the initial term structure of the zero-coupon bonds is positive and non-increasing, this is always ensured.

4.2. Ensuring an Arbitrage-Free Environment

The first important issue to be considered is the question if the MSJD-extension to the LMM is free of arbitrage possibilities. This is indeed the case, as the following proposition demonstrates. We prove that no-arbitrage holds by showing that the MSJD-extension to the LMM can be embedded into the generalized HJM framework of Björk et al. (1997), for which no-arbitrage conditions are known.[3] We partly follow a construction similar to Eberlein and Özkan (2005), before completing the proof in the spirit of Brace et al. (1997). Absence of arbitrage then follows from the no-arbitrage condition in the instantaneous forward rate model.

We consider a slightly modified version of the generalized HJM model of Björk et al. (1997) and Björk et al. (1997). In a Markov-switching version of their model, it is assumed that for a given measure $\mathcal{Q}$, the dynamics of the instantaneous forward rates follow

$$
\begin{aligned}
(4.3) \qquad df(t,T) &= \alpha^*(t, X_{t-}, T)\, dt + \varsigma^*(t, X_{t-}, T)\, dW^{\mathcal{Q}}(t) \\
&\quad + \int_{\mathbb{R}^d} \gamma^*(t, X_{t-}, T, z)\, [\mu - \nu^{\mathcal{Q}}_{X_{t-}}]\,(dt, dz)\,, \qquad \forall\, t \leq T,
\end{aligned}
$$

where $W^{\mathcal{Q}}$ is a $\mathcal{Q}$-Brownian motion and $\nu^{\mathcal{Q}}_{X_{t-}}(dt, dz)$ the predictable $\mathcal{Q}$-compensator of μ. Furthermore, it is assumed that $\nu^{\mathcal{Q}}_{X_{t-}}(dt, dz) = K^{\mathcal{Q}}(t, X_{t-}, dz)\, dt$, with $K^{\mathcal{Q}}(t, X_{t-}, A)\, dt$ predictable for all $\mathcal{A} \in \mathcal{E}$. The functions α^*, ς^* and γ^* are assumed to be deterministic conditional on X_{t-} and satisfy all necessary integrability conditions as postulated in Björk et al. (1997). By comparing the dynamics of the instantaneous forward rates with those of the bond prices, Björk et al. (1997) show that $\mathcal{Q}$ is a martingale

[3] Observe that, given the presence of a state price density, the existence of an arbitrage-free framework where all LIBOR rates have dynamics according to (4.2) follows immediately from the general construction for semimartingales in Jamshidian (1999). However, given the complexity of his proofs, one may also follow a different type of proof that allows for a thorough introduction and understanding of the model.

measure if and only if the no-arbitrage condition

$$\begin{aligned}\alpha_i\left(t, X_{t-}\right) = &-\frac{1}{2}\left\|\varsigma_i\left(t, X_{t-}\right)\right\|^2 \\ &- \int_{\mathbb{R}^k}\left(e^{\gamma_i(t,X_{t-},z)} - 1 - \gamma_i\left(t, X_{t-}, z\right)\right) K^{\mathcal{Q}}\left(t, X_{t-}, dz\right),\end{aligned} \tag{4.4}$$

holds, where

$$\begin{aligned}\alpha_i\left(t, X_{t-}\right) &:= -\int_t^{T_i} \alpha^*\left(t, X_{t-}, u\right) du, \qquad \varsigma_i\left(t, X_{t-}\right) := -\int_t^{T_i} \varsigma^*\left(t, X_{t-}, u\right) du, \\ \gamma_i\left(t, X_{t-}, z\right) &:= -\int_t^{T_i} \gamma^*\left(t, X_{t-}, u, z\right) du.\end{aligned} \tag{4.5}$$

In order to prove that the LIBOR market model can indeed be embedded in the extended HJM model, the particular relationship between instantaneous and simple forward rates is exploited,

$$L_i(t) = \frac{1}{\delta}\left[\frac{B\left(t, T_i\right)}{B(t, T_{i+1})} - 1\right] = \frac{1}{\delta}\left[\exp\left(\int_{T_i}^{T_{i+1}} f(t,s)ds\right) - 1\right]. \tag{4.6}$$

Even though the calculations are occasionally cumbersome, the use of the no-arbitrage condition (4.4), property (4.6) and measure changes then eventually yields the following proposition:

Proposition 4.1 (Embedding the MSJD LMM Model into the Regime-Switching HJM-Model)**.**

1. *Based on* (4.3)*, the* $\mathcal{Q}$*–dynamics of* L_i*,* $i = 1, \ldots, N-1$*, follow the dynamics*

$$\begin{aligned}&\frac{\delta}{1 + \delta L_i\left(t-\right)} \cdot dL_i\left(t\right) \\ &\quad = \left(\varsigma_{i+1}\left(t, X_{t-}\right) - \varsigma_i\left(t, X_{t-}\right)\right)' \varsigma_{i+1}\left(t, X_{t-}\right) dt \\ &\quad + \int_{\mathbb{R}^k}\left(e^{\gamma_i(t,X_{t-},z) - \gamma_{i+1}(t,X_{t-},z)} - 1 + e^{\gamma_{i+1}(t,X_{t-},z)} - e^{\gamma_i(t,X_{t-},z)}\right) \nu^{\mathcal{Q}}_{X_{t-}}\left(dt, dx\right) \\ &\quad + \left(\varsigma_i\left(t, X_{t-}\right) - \varsigma_{i+1}\left(t, X_{t-}\right)\right)' dW^{\mathcal{Q}}\left(t\right) \\ &\quad + \int_{\mathbb{R}^k}\left(e^{\gamma_i(t,X_{t-},z) - \gamma_{i+1}(t,X_{t-},z)} - 1\right)\left(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\right)\left(dt, dz\right).\end{aligned} \tag{4.7}$$

2. *The $\mathcal{Q}^{i+1}$–dynamics of L_i are given as*

$$\frac{\delta}{1+\delta L_i(t-)} \cdot dL_i(t) = \left(\varsigma_i(t, X_{t-}) - \varsigma_{i+1}(t, X_{t-})\right)' dW^{i+1}(t) + \int_{\mathbb{R}^k} \left(e^{\gamma_i(t, X_{t-}, z) - \gamma_{i+1}(t, X_{t-}, z)} - 1\right) \left(\mu - \nu^{i+1}_{X_{t-}}\right)(dt, dz). \tag{4.8}$$

3. *The $\mathcal{Q}^{i+1}$–dynamics of L_i take the form*

$$\frac{dL_i(t)}{L_i(t-)} = \sigma_i(t, X_{t-})' dW^{i+1}(t) + \int_{\mathbb{R}^k} \psi_i(t, X_{t-}, z) \left(\mu - \nu^{i+1}_{X_{t-}}\right)(dt, dz), \tag{4.9}$$

with W^{i+1} a $\mathcal{Q}^{i+1}$-Brownian motion, ν^{i+1} the $\mathcal{Q}^{i+1}$-compensator of μ and σ_i and ψ_i appropriately defined.

Proof. The proof is based on a three-step procedure. Details are elaborated in Appendix A. □

4.3. The Measure Changes and their Consequences

In a next step, we consider how the dynamics of LIBOR rates with different maturities can be interrelated. This is in particular important for the pricing of interest rate products, as these oftentimes depend not only on one, but rather multiple LIBOR rates of different maturities. In the following, the consequences of measure changes for the Wiener processes and the compensators as well as the Markov chain and its generator are investigated.

Consequences of Measure Changes on Wiener Process and Compensator Measure

The Radon-Nikodým derivative $\eta_{i+2,i+1}(t)$ associated with a measure change from $\mathcal{Q}^{i+2}$ to $\mathcal{Q}^{i+1}$ is given as

$$\eta_{i+2,i+1}(t) := \left.\frac{d\mathcal{Q}^{i+1}}{d\mathcal{Q}^{i+2}}\right|_{\mathcal{F}_t} = \frac{B(0,T_{i+2})}{B(0,T_{i+1})} \cdot \frac{B(t,T_{i+1})}{B(t,T_{i+2})} = \frac{B(0,T_{i+2})}{B(0,T_{i+1})} \cdot \left[1 + \delta L_{i+1}(t)\right]$$

and, following from (4.2), has dynamics

$$d\eta_{i+2,i+1}(t) = \eta_{i+2,i+1}(t-)\frac{\delta L_{i+1}(t-)}{1+\delta L_{i+1}(t-)} \times \left[\sigma_{i+1}(t,X_{t-})' dW^{i+2}(t) + \int_{\mathbb{R}^k} \psi_{i+1}(t,X_{t-},z)\left(\mu - \nu^{i+2}_{X_{t-}}\right)(dt,dz)\right] \tag{4.10}$$

for $i = N-2,\ldots,1$. Making use of Girsanov's Theorem 2.3, it follows iteratively that the $\mathcal{Q}^{i+1}$-Wiener process W^{i+1} and the $\mathcal{Q}^{i+1}$-compensator $\nu^{i+1}_{X_{t-}}$ of μ may be written with respect to their counterparts under the terminal measure $\mathcal{Q}^N$ as

$$W^{i+1}(t) = -\int_0^t \sum_{j=i+1}^{N-1} \frac{\delta L_j(s-)}{1+\delta L_j(s-)}\sigma_j(s,X_{s-})\,dt + W^N(t), \tag{4.11}$$

$$\nu^{i+1}_{X_{t-}}(dt,dz) = \prod_{j=i+1}^{N-1}\left(1+\frac{\delta L_j(t-)\,\psi_j(t,X_{t-},z)}{1+\delta L_j(t-)}\right)\nu^N_{X_{t-}}(dt,dz). \tag{4.12}$$

Observe that Girsanov's Theorem 2.3 even allows to make a statement about how jump intensity and the distribution of the markers change. Inserting (4.11) and (4.12) into (4.2), the dynamics of each LIBOR rate L_i, $i = 1,\ldots,M$ can be expressed in terms of $\mathcal{Q}^N$,

$$\begin{aligned}\frac{dL_i(t)}{L_i(t-)} =& -\sum_{j=i+1}^{N-1}\frac{\delta L_j(t-)}{1+\delta L_j(t-)}\sigma_i(t,X_{t-})'\sigma_j(t,X_{t-})\,dt + \sigma_i(t,X_{t-})'\,dW^N(t) \\ &- \int_{\mathbb{R}^k}\psi_i(t,X_{t-},z)\Big(\prod_{j=i+1}^{N-1}\big(1+\frac{\delta L_j(t-)}{1+L_j(t-)}\cdot\psi_j(t,X_{t-},z)\big)-1\Big)\nu^N_{X_{t-}}(dt,dz) \\ &+ \int_{\mathbb{R}^k}\psi_i(t,X_{t-},z)\left(\mu-\nu^N_{X_{t-}}\right)(dt,dz),\end{aligned} \tag{4.13}$$

with W^N being a $\mathcal{Q}^N$ Brownian motion and $\nu^N_{X_{t-}}$ the $\mathcal{Q}^N$-compensator of μ.

Equations (4.11) and (4.12) may now be used to justify the rather strict requirements (II) and (III):

Remark 4.2 (Measure Changes and Model Assumptions (II) and (III))**.**
Assumption (II) can indeed only be specified for the terminal measure $\mathcal{Q}^N$ *as this feature cannot be transferred from the terminal measure to any other forward measure* $\mathcal{Q}^2,\ldots,\mathcal{Q}^{N-1}$*. This is due to the fact that both* $W^{i+1}(t)$ *and* $\nu^{i+1}_{X_{t-}}$ *in* (4.11) *and* (4.12) *contain terms* $L_{i+1},\ldots,L_{N-1}$*, which in turn*

contain integral terms involving W^N *and* $\nu^N_{X_{t-}}$.
A similar problem arises for Requirement (III), where the terminal compensator $\nu^N_{X_{t-}}$ *was defined to be the compensator of a time-homogeneous Markov switching measure associated with a marked Poisson process, with Markov-switching jump intensity* $\lambda^N(X_{t-})$ *and marker distribution* $k^N(X_{t-}, dz)$. *This means, that conditional on the state of the Markov chain, the compensator is deterministic. Observing* (4.12), *this property is also not preserved under the measure change to any other measure* $\mathcal{Q}^{i+1}$, $i = 1, \dots, N-2$, *due to the factors* $\delta L_i / (1 + \delta L_i)$, $j = i+1, \dots, N$.

Both points mentioned turn out to be particular inconvenient when it comes to pricing of caps/caplets and swaptions in Section 6. The problem can, however, be circumvented by approximating (4.11) and (4.12) via freezing the L_j's at 0, as proposed by Belomestny and Schoenmakers (2011). Then,

$$dW^{i+1}(t) \approx d\widetilde{W}^{i+1}(t) = \sum_{j=i+1}^{N-1} \frac{\delta L_j(0)}{1 + \delta L_j(0)} \sigma_j(t, X_{t-})\, dt + dW^N(t), \tag{4.14}$$

$$\nu^{i+1}_{X_{t-}}(dt, dz) \approx \widetilde{\nu}^{i+1}_{X_{t-}}(dt, dz) := \prod_{j=i+1}^{N-1} \left(1 + \frac{\delta L_j(0)\, \psi_j(t, X_{t-}, z)}{1 + \delta L_j(0)}\right) \nu^N_{X_{t-}}(dt, dz), \tag{4.15}$$

the $\widetilde{\nu}^{i+1}_{X_{t-}}$ is state-dependent deterministic and the independence between compensator and Wiener process is preserved.

The Consequences of Measure Changes on the Markov Chain

It is yet to be checked how measure changes influences the Markov chain. So far, we have used the simple, no-index notation $\mathcal{A}$ for the infinitesimal generator of X. The following proposition demonstrates that this is in fact a sensible notation, as the infinitesimal generator is not affected by changes between forward measures:

Proposition 4.3 (Markov Chain under the Measure Change)**.** *The measure change from* $\mathcal{Q}^{i+1}$ *to* $\mathcal{Q}^i$ *has no influence on the infinitesimal generator of the Markov chain* X.

Proof. For the purpose of clarity, we write $\mathcal{A}^i$ as the infinitesimal generator of X under $\mathcal{Q}^i$. By definition of the infinitesimal generator under the T_i-forward measure $\mathcal{Q}^i$,

$$\mathcal{A}^i f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^{\mathcal{Q}^i}\left[f(X_t) | X_0 = x\right] - f(x)}{t} =: \lim_{t \downarrow 0} \frac{\mathbb{E}^{\mathcal{Q}^i}_x\left[f(X_t)\right] - f(x)}{t} \tag{4.16}$$

for any bounded, Borel-measurable function f, for which this limit exists. Accordingly, $\mathcal{A}^{i+1}$ is the infinitesimal generator of X with respect to $\mathcal{Q}^{i+1}$. We rewrite (4.16) as

$$\begin{aligned}
\mathcal{A}^i f(x) &= \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathcal{Q}^i}\left[f(X_t)\right] - \mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t)\right] + \mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t)\right] - f(x)}{t} \\
&= \underbrace{\lim_{t\downarrow 0}\left[\frac{\mathbb{E}_x^{\mathcal{Q}^i}\left[f(X_t)\right] - \mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t)\right]}{t}\right]}_{R_i(X_t)} + \mathcal{A}^{i+1} f(x).
\end{aligned} \tag{4.17}$$

In order to show that the measure change does not have any influence on the infinitesimal generator, it suffices to show that $R_i(X_t) = 0$. Using the Radon-Nikodým derivative $\eta_{i+1,i}(t) := d\mathcal{Q}^i/d\mathcal{Q}^{i+1}|_{\mathcal{F}_t}$, we may write $R_i(X_t)$ in terms of the $\mathcal{Q}^{i+1}$-expectation:

$$R_i(X_t) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[\frac{d\mathcal{Q}^i}{d\mathcal{Q}^{i+1}}|_{\mathcal{F}_t} \cdot f(X_t)\right] - \mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t)\right]}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[\left(\eta_{i+1,i}(t) - 1\right) f(X_t)\right]}{t}.$$

The tower property for expectations yields, conditional on X_t,

$$R_i(X_t) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t)\, \mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[\left(\eta_{i+1,i}(t) - 1\right) \middle| X_t\right]\right]}{t},$$

using that $f(X_t)$ is, naturally, measurable with respect to X_t. The solution to (4.10) is by the Doléans-Dade exponential given according to expression (2.4),

$$\begin{aligned}
\eta_{i+1,i}(t) = \exp\Big(&-\frac{1}{2}\int_0^t \left\| \frac{\delta L_i(s-)}{1+\delta L_i(s-)} \sigma_i(s, X_{s-}) \right\|^2 ds \\
&+ \int_0^t \frac{\delta L_i(s-)}{1+\delta L_i(s-)} \sigma_i(s, X_{s-})' \, dW^{i+1}(s) \\
&+ \int_0^t \int_{\mathbb{R}^d} \ln\left(1 + \frac{\delta L_i(s-)}{1+\delta L_i(s-)} \psi_i(s, X_{s-}, z)\right) \left(\mu - \nu_{X_{t-}}^{i+1}\right)(ds, dz) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Big[\ln\left(1 + \frac{\delta L_i(s-)}{1+\delta L_i(s-)} \psi_i(s, X_{s-}, z)\right) \\
&\qquad - \frac{\delta L_i(s-)}{1+\delta L_i(s-)} \psi_i(s, X_{s-}, z) \Big] \nu_{X_{t-}}^{i+1}(ds, dz) \Big).
\end{aligned} \tag{4.18}$$

In particular, $\eta_{i+1,i}(t)$ is of exponential integral form and for all possible values of the finite state space Markov chain X at t, $\mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[\left(\eta_{i+1,i}(t) - 1\right) \middle| X_t = j\right] = 0$, as $\eta_{i+1,i}(t)$ is a $\mathcal{Q}^{i+1}$-martingale with starting value $\eta_{i+1,i}(0) = 1$. Consequently,

$$R_i(X_t) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x^{\mathcal{Q}^{i+1}}\left[f(X_t) \cdot 0\right]}{t} = 0. \tag{4.19}$$

Inserting (4.19) into (4.17) yields $\mathcal{A}^i = \mathcal{A}^{i+1}$ and the proposition is proved. □

Remark 4.4 (The Infinitesimal Generator and the Change from $\mathcal{Q}$ to $\mathcal{Q}^{i+1}$).
The same statement about the invariability of the generator of the Markov chain is true when changing from the spot martingale measure $\mathcal{Q}$ to $\mathcal{Q}^{i+1}$. As in the previous proposition, this follows from the exponential integral-type representation of $d\mathcal{Q}^{i+1}/d\mathcal{Q}|_{\mathcal{F}_t}$, similar to (4.18).

5. Embedding the Swap Dynamics into the LMM Extension

As it was mentioned in Section 3.3, there are two possible ways of how swaptions can be evaluated. Either, swap rates are modeled as martingale processes under the corresponding swap measure and then swaptions are priced analytically with the Black (1976)-type formula (3.8), or the swap dynamics are suitably embedded into the LMM and swaption prices are derived based on appropriate approximations. As the former case can be done completely analogous to Section 4, it shall not be considered here. Instead, we concentrate on the second case.

5.1. Changing to the Swap Measure $\mathcal{Q}^{\alpha,\beta}$

The first step is to examine how measure changes from the terminal measure $\mathcal{Q}^N$ (or any other forward measure) to the swap measure $\mathcal{Q}^{\alpha,\beta}$ affect Wiener processes and compensator measures:

Theorem 5.1 (Measure Change from $\mathcal{Q}^N$ to $\mathcal{Q}^{\alpha,\beta}$).
Wiener processes W^N and $W^{\alpha,\beta}$ and compensator measures $\nu^N_{X_{t-}}$ and $\nu^{\alpha,\beta}_{X_{t-}}$ under the corresponding

measures $\mathcal{Q}^N$ and $\mathcal{Q}^{\alpha,\beta}$ can be related via

$$dW^{\alpha,\beta}(t) = dW^N(t) - \sum_{i=\alpha}^{\beta-1} z_{i+1,\alpha,\beta}(t) \sum_{j=i+1}^{N-1} \frac{\delta L_j(t-)}{1+\delta L_j(t-)} \sigma_j(t, X_t)\, dt,$$

$$\nu_{X_{t-}}^{\alpha,\beta}(dt,dz) = \left[\Big(\sum_{i=\alpha}^{\beta-1} z_{i+1,\alpha,\beta}(t)\Big(\prod_{j=i+1}^{N-1}\left[1+\frac{\delta L_j(t-)\,\psi_j(t,X_{t-},z)}{1+\delta L_j(t-)}\right]-1\Big)\Big)+1\right] \times \nu_{X_{t-}}^N(dt,dz),$$

where $z_{i+1,\alpha,\beta}(t) \coloneqq \delta B(t,T_{i+1})/C_{\alpha,\beta}(t)$.

Proof. For the proof, see Appendix B. □

The intensities and jump distribution furthermore relate as follows:

$$\lambda^{\alpha,\beta}(t,X_{t-}) = \lambda^N(t,X_{t-}) \int_{\mathbb{R}^k} \left[\Big(\sum_{i=\alpha}^{\beta-1} z_{i+1,\alpha,\beta}(t) \times \Big(\prod_{j=i+1}^{N-1}\Big(1+\frac{\delta L_j(t-)\,\psi_j(t,X_{t-},z)}{1+\delta L_j(t-)}\Big)-1\Big)\Big)+1\right] k^N(X_{t-},dz), \tag{5.1}$$

$$k^{\alpha,\beta}(X_{t-},dz) = k^N(X_{t-},dz) \times \frac{\left[\Big(\sum_{i=\alpha}^{\beta-1} z_{i+1,\alpha,\beta}(t)\Big(\prod_{j=i+1}^{N-1}\Big(1+\frac{\delta L_j(t-)\psi_j(t,X_{t-},z)}{1+\delta L_j(t-)}\Big)-1\Big)\Big)+1\right]}{\int_{\mathbb{R}^k}\left[\Big(\sum_{i=\alpha}^{\beta-1} z_{i+1,\alpha,\beta}(t)\Big(\prod_{j=i+1}^{N-1}\Big(1+\frac{\delta L_j(t-)\psi_j(t,X_{t-},z)}{1+\delta L_j(t-)}\Big)-1\Big)\Big)+1\right] k^N(X_{t-},dz)}. \tag{5.2}$$

5.2. Approximative Swap Dynamics

Following Rebonato (2002), one possibility of how swap rate dynamics could be approximated is to rewrite (3.3) as

$$S_{\alpha,\beta}(t) = w_\alpha(t) L_\alpha(t) + \ldots + w_{\beta-1}(t) L_{\beta-1}(t),$$

where the "weights" $w_j(t), j = \alpha,\ldots,\beta-1$ are defined as $w_j(t) = \delta \cdot B(t,T_{j+1})/C_{\alpha,\beta}(t)$. One could then freeze the weights at $t = 0$, such that

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} w_i(0) L_i(t). \tag{5.3}$$

Differentiation on both sides would then yield an approximation of the swap dynamics. This is obviously a rather crude approximation, and we walk along a different path. Instead of first freezing the weights and then applying Itō's formula to (5.3), we follow the idea initially proposed by Andersen and Andreasen (2000) to first derive and *then* freeze any weights. Andersen and Brotherton-Ratcliffe (2005) apply this idea to a stochastic volatility environment without jumps, but the approach is just as applicable to the MSJD case. The idea yields the following proposition, where the swap rate dynamics are expressed in a combination of terms involving bond prices, the annuity and the coefficient functions in the LIBOR rate dynamics:

Proposition 5.2 (Swap Rate Dynamics)**.**
Using the LIBOR rate coefficient functions, the swap rate dynamics take the form

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t-)} = \sum_{j=\alpha}^{\beta-1} x_j(t)\,\sigma_j(t, X_{t-})'\,dW^{\alpha,\beta}(t) + \int_{\mathbb{R}^k} \psi_{\alpha,\beta}(t, X_{t-}, z)\left(\mu - \nu^{\alpha,\beta}_{X_{t-}}\right)(dt, dz), \tag{5.4}$$

where

$$x_j(t) = \frac{\delta L_j(t-)}{1+\delta L_j(t)}\left(\frac{B(t,T_\beta)}{B(t,T_\alpha)-B(t,T_\beta)} + \frac{1}{C_{\alpha,\beta}(t)}\cdot\delta\sum_{k=j}^{\beta-1} B(t,T_{k+1})\right), \text{ and}$$

$$\psi_{\alpha,\beta}\big(t, X_{t-}, z\big) = \frac{1}{S_{\alpha,\beta}(t-)}\frac{1-\prod_{i=\alpha}^{\beta-1}\frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}}{\delta\sum_{k=\alpha}^{\beta-1}\prod_{i=\alpha}^{k}\frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}} - 1. \tag{5.5}$$

Proof. For the proof, see Appendix C. □

For the approximation of the dynamics, the weights x_j are frozen at $t = 0$. Defining $\widetilde{\psi}$ as

$$\widetilde{\psi}_{\alpha,\beta}(t, X_{t-}, z) = \frac{1}{S_{\alpha,\beta}(0)}\frac{1-\prod_{i=\alpha}^{\beta-1}\frac{1}{1+\delta L_i(0)[1++\psi_i(t,X_{t-},z)]}}{\delta\sum_{k=\alpha}^{\beta-1}\prod_{i=\alpha}^{k}\frac{1}{1+\delta L_i(0)[1+\psi_i(t,X_{t-},z)]}} - 1,$$

the approximate dynamics are given as

$$\frac{d\widetilde{S}_{\alpha,\beta}(t)}{\widetilde{S}_{\alpha,\beta}(t-)} = \sum_{j=\alpha}^{\beta-1} x_j(0)\,\sigma_j(t, X_{t-})'\,dW^{\alpha,\beta}(t) + \int_{\mathbb{R}^k} \widetilde{\psi}_{\alpha,\beta}(t, X_{t-}, z)\left(\mu - \nu_{X_{t-}}^{\alpha,\beta}\right)(dt, dz)\,. \tag{5.6}$$

Intensity and jump size distribution may also be approximated based on (5.1) and (5.2).

6. Pricing in the MSJD Framework

Given the dynamics of the MSJD extension to the LMM in Section 4 and the embedding of swap rates into the model in Section 5, we now turn towards the pricing of caplets/caps and swaptions. For convenience, recall that each LIBOR rate follows dynamics

$$\frac{dL_i(t)}{L_i(t-)} = \sigma_i(t, X_{t-})'\,dW^{i+1}(t) + \int_{\mathbb{R}^k} \psi_i(t, X_{t-}, z)\left(\mu - \nu_{X_{t-}}^{i+1}\right)(dt, dz)\,,$$

where σ_i and ψ_i are state-dependent functions, W^{i+1} is a d-dimensional $\mathcal{Q}^{i+1}$-Brownian motion and $\nu_{X_{t-}}^{i+1}$ is the $\mathcal{Q}^{i+1}$-compensator measure of the random jump measure μ. There are different possibilities of how volatilities, jump functions and compensators in the MSJD framework can be specified. Since the calibration of the model will involve the fitting of parameters related to a wide range of LIBOR rates with different maturities, we limit in a first step the dimension of Wiener process and jump space to $d = 1$ and $k = 1$.[4]

Like in the introductory Section 3.3 on the log-normal LMM, the price of a caplet is given as

$$\text{Caplet}_i(t) = \delta \cdot B(t, T_{i+1}) \cdot \mathbb{E}_{\mathcal{Q}^{i+1}}\left[(L_i(T_i) - K)^+ \middle| \mathcal{F}_t\right]. \tag{6.1}$$

We employ the Laplace transform in order to further evaluate (6.1) in the MSJD specification. For simplicity, let $t = 0$. The price of a caplet on the i-th LIBOR rate with strike K at time 0 may be

[4] At least in the case, where no jumps are considered, this is a justifiable assumption, when it comes to caplet pricing. Similar to pricing formulas (3.6) and (3.7) developed in the log-normal LMM, our considerations show that prices depend only on the norm of σ_i, that is $\|\sigma_i(t, X_{t-})\|$, and not on $\sigma_i(t, X_{t-})$ itself. As underlined, e.g., by Filipovic (2009), p. 213, there is thus no gain in flexibility for caplet pricing by introducing additional dimension into the model. Note, nonetheless, that this is no longer true for swaption pricing.

rewritten as

$$\text{Caplet}_i(0) = \delta \cdot B(0, T_{i+1}) \cdot K \cdot \mathbb{E}_{\mathcal{Q}^{i+1}}\Big[\big(e^{Y_i(T_i)-k_i} - 1\big)^+\Big], \tag{6.2}$$

with $Y_i(t) := \ln(L_i(t)/L_i(0)) = \ln(L_i(t)) - \ln(L_i(0))$ and $k_i = \ln(K/L_i(0))$. Using the Laplace transform (see Raible (2000)), the caplet's price is given as

$$\text{Caplet}_i(0) = Ke^{xk_i}\delta B(0, T_{i+1}) \frac{1}{\pi}\int_0^\infty \text{Re}\left(e^{iuk_i}\frac{\phi_i(ix-u, T_i)}{x^2 + x - u^2 + iu(2x+1)}\right)du, \tag{6.3}$$

for some $x < -1$, with $\phi_i(u, t) = \mathbb{E}_{\mathcal{Q}^{i+1}}[\exp(iuY_i(t))]$ denoting the characteristic function of Y_i under the corresponding T_{i+1}–forward measure. For the evaluation of (6.3), it is necessary to determine a closed-form expression for $\phi_i(ix-u, T_i)$ for all $1 \leq i \leq N-1$.

6.1. Determining the Characteristic Function of Y_{N-1}

The crucial point in the pricing of caplets is to start with the LIBOR rate $L_{N-1}(t)$ of longest maturity T_{N-1} whose dynamics are given as

$$\frac{dL_{N-1}(t)}{L_{N-1}(t-)} = \sigma_{N-1}(t, X_{t-})\,dW^N(t) + \int_{\mathbb{R}} \psi_{N-1}(t, X_{t-}, z)\left(\mu - \nu^N_{X_{t-}}\right)(dt, dz). \tag{6.4}$$

As observed before, this LIBOR rate takes a special role among the other rates, because it is the only rate, where the jump distribution is assumed to be known and the intensity is constant when conditioned on the state of the underlying Markov chain,

$$\nu^N_{X_{t-}}(dt, dz) = \lambda^N(X_{t-})\,k^N(X_{t-}, dz)\,dt.$$

The solution to the SDE (6.4) is given by the Doléans-Dade exponential

$$\begin{aligned} L_{N-1}(t) = L_{N-1}(0) \cdot \exp\Big(-\frac{1}{2}\int_0^t &\big[\sigma^2_{N-1}(s, X_{s-})ds + \int_0^t \sigma_{N-1}(s, X_{t-})dW^N(s) \\ &+ \int_0^t\int_{\mathbb{R}} \ln(1 + \psi_{N-1}(s, X_{s-}, z))\mu(ds, dz) \\ &- \int_0^t\int_{\mathbb{R}} \psi_{N-1}(s, X_{s-}, z)\nu^N_{X_{s-}}(ds, dz)\big]\Big), \end{aligned}$$

which immediately implies the dynamics of $Y_{N-1}(t) = \ln(L_{N-1}(t)/L_{N-1}(0))$. In particular, for X being in a fixed state, say $X_{t-} \equiv e_j$, $Y_{N-1}(t,j) := Y_{N-1}(t)\,|_{X_{t-}=e_j}$ has dynamics

$$\begin{aligned} dY_{N-1}(t,j) = &-\frac{1}{2}\sigma^2_{N-1}(t,e_j)\,dt + \sigma_{N-1}(t,e_j)\,dW^N(t) \\ &- \int_{\mathbb{R}} \psi_{N-1}(t,e_j,z)\,\nu_j^N(dt,dz) + \int_{\mathbb{R}} \ln(1+\psi_{N-1}(t,e_j,z))\,\mu(dt,dz)\,. \end{aligned} \tag{6.5}$$

From this, the characteristic function of $Y_{N-1}(t,j)$ can be easily derived:

Proposition 6.1 (Characteristic Function of $Y_{N-1}(.,j)$**).**
The characteristic function $\phi_{N-1}(u,t,j) = \mathbb{E}_{Q^N}[\exp(iuY_{N-1}(t,j))]$ of $Y_{N-1}(.,j)$ is given as

$$\phi_{N-1}(u,t,j) = \exp\left(\int_0^t \zeta_{N-1}(s,e_j,u)\,ds\right),$$

with

$$\begin{aligned} \zeta_{N-1}(s,e_j,u) := &-\frac{u^2}{2}\sigma^2_{N-1}(s,e_j) - \frac{1}{2}iu\sigma^2_{N-1}(s,e_j) - iu\lambda^N(e_j)\int_{\mathbb{R}} \psi_{N-1}(s,e_j,z)\,k^N(e_j,dz) \\ &+ \lambda^N(e_j)\int_{\mathbb{R}} [\exp(iu\ln(\psi_{N-1}(s,e_j,z)+1)) - 1]\,k^N(e_j,dz)\,. \end{aligned}$$

Proof. Since the first and third term in (6.5) are deterministic, it suffices to analyze the second and fourth term, which are, by assumption (II), independent. The claim then follows from the general form of characteristic functions for Brownian motions and marked Poisson processes. □

Proposition 6.1 implies that the characteristic function $\phi_{N-1}(u,t)$ of Y_{N-1} may be rewritten as

$$\begin{aligned} \phi_{N-1}(u,t) &= \mathbb{E}_{Q^N}\big[\exp\big(iuY_{N-1}(t)\big)\big] = \mathbb{E}_{Q^N}\left[\mathbb{E}_{Q^N}\left[\exp(iuY_{N-1}(t))\middle|\mathcal{F}_t^X\right]\right] \\ &= \mathbb{E}_{Q^N}\Big[\exp\Big(\int_0^t \Big[-\frac{u^2}{2}\sigma^2_{N-1}(s,X_{s-}) - \frac{1}{2}iu\sigma^2_{N-1}(s,X_{s-}) \\ &\qquad - iu\lambda^N(X_{s-})\int_{\mathbb{R}} \psi_{N-1}(s,X_{s-},z)\,k^N(X_{s-},dz) \\ &\qquad + \lambda^N(X_{s-})\int_{\mathbb{R}} [\exp(iu\ln(\psi_{N-1}(s,X_{s-},z)+1)) - 1]\,k^N(X_{s-},dz)\Big]ds\Big)\Big], \end{aligned}$$

by the law of iterated expectation. Let

$$
\begin{aligned}
Z_{N-1}(t,u) := \exp\Big(\int_0^t \Big[&- \frac{u^2}{2}\sigma_{N-1}^2(s,X_{s-}) - \frac{1}{2} iu\sigma_{N-1}^2(s,X_{s-}) \\
&- iu\lambda^N(X_{s-}) \int_{\mathbb{R}} \psi_{N-1}(s,X_{s-},z)\, k^N(X_{s-},dz) \\
&+ \lambda^N(X_{s-}) \int_{\mathbb{R}} \left[\exp\left(iu\ln\left(\psi_{N-1}(s,X_{s-},z)+1\right)\right) - 1\right] k^N(X_{s-},dz) \Big] ds \Big).
\end{aligned}
$$

In the special case that σ_{N-1} and ψ_{N-1} are regime-dependent constant, $\phi_{N-1}(u,t) = \mathbb{E}_{Q^N}[Z_{N-1}(t,u)]$ may then be evaluated as follows:

Proposition 6.2 (Characteristic Function of Y_{N-1})**.**
Let X be a Markov chain with infinitesimal generator $\mathcal{A}$ taking its values in the M-dimensional state space $E = \{e_1, \ldots, e_M\}$. Furthermore, let $\sigma_{N-1}(t, X_{t-}) \equiv \sigma_{N-1}(X_{t-})$ and $\psi_{N-1}(t, X_{t-}, u) \equiv \psi_{N-1}(X_{t-}, u)$ of each state be non-time-dependent. Then, the characteristic function $\phi_{N-1}(u,t) = \mathbb{E}_{Q^N}[\exp(iuY_{N-1}(t))]$ of Y_{N-1} is given as

$$
\phi_{N-1}(u,t) = \langle 1_M, \exp\left(\mathcal{C}_{N-1}(u)\cdot t\right) X_0\rangle \tag{6.6}
$$

where $1_M \in \mathbb{R}^M$ is the vector consisting only of ones, $\langle .,. \rangle$ the Euclidean scalar product in $\mathbb{R}^M$ and $\mathcal{C}_{N-1}(u)$ given as

$$
\mathcal{C}_{N-1}(u) = \mathcal{A}' + \textit{diag}\left(\zeta_{N-1}(e_1,u), \ldots, \zeta_{N-1}(e_M,u)\right),
$$

with $\zeta_{N-1}(e_j,u) \equiv \zeta_{N-1}(t,e_j,u)$ given as in Proposition 6.1.

Proof. The proof is in parts based on the considerations in Elliott and Valchev (2004). Set $G_t = X_t \cdot Z_{N-1}(t,u)$. Using the semimartingale decomposition (4.1), $X_t = X_0 + \int_0^t \mathcal{A}' X_s ds + M_t$ of X, it follows by integration by parts that

$$
\begin{aligned}
G_t &= G_0 + \int_0^t Z_{N-1}(s-)\left[\mathcal{A}' X_{s-} ds + dM_s\right] + \int_0^t X_{s-} dZ_{N-1}(s) \\
&= G_0 + \int_0^t \mathcal{A}' G_{s-} ds + \int_0^t (\ldots)\, dM_s + \int_0^t X_s dZ_{N-1}(s).
\end{aligned}
$$

Hence,

$$\begin{aligned}G_t = G_0 + \int_0^t \mathcal{A}' G_s ds + \int_0^t (\dots)\, dM_s \\ + \int_0^t \Big[-\frac{u^2}{2}\sigma_{N-1}^2 (X_{s-}) - \frac{1}{2} iu\sigma_{N-1}^2 (X_{s-}) \\ - iu\lambda^N (X_{s-}) \int_{\mathbb{R}} \psi_{N-1}(X_{s-}, z)\, k(X_{s-}, dz) \\ + \lambda^N (X_{s-}) \int_{\mathbb{R}} [\exp(iu \ln(\psi_{N-1}(X_{s-}, z) + 1)) - 1]\, k(X_{s-}, dz) \Big] X_s Z_{N-1}(s) ds.\end{aligned} \tag{6.7}$$

Observe that $G_0 = X_0$. We set $\zeta_{N-1}(u) := (\zeta_{N-1}(e_1, u), \dots, \zeta_{N-1}(e_M, u))$, where $\zeta_{N-1}(e_j, u)$, $j = 1, \dots, M$ are defined as above, according to Proposition 6.1. Then, (6.7) reads

$$\begin{aligned}G_t &= X_0 + \int_0^t \mathcal{A}' G_s ds + \int_0^t (\dots)\, dM_s + \int_0^t \big\langle \zeta_{N-1}(u), X_s \big\rangle X_s Z_{N-1}(s) ds \\ &= X_0 + \int_0^t \left[\mathcal{A}' + \text{diag}(\zeta_{N-1}(e_1, u), \dots, \zeta_{N-1}(e_M, u))\right] G_s ds + \int_0^t (\dots)\, dM_s.\end{aligned}$$

M is a martingale with respect to $\mathbb{F}^X$. Hence, taking the expectation on the previous formula yields, under application of Fubini's Theorem,

$$\mathbb{E}_{\mathcal{Q}^{i+1}}[G_t] = X_0 + \int_0^t \underbrace{\left[\mathcal{A}' + \text{diag}(\zeta_{N-1}(e_1, u), \dots, \zeta_{N-1}(e_M, u))\right]}_{=:B} \mathbb{E}_{\mathcal{Q}^{i+1}}[G_s]\, ds + 0. \tag{6.8}$$

For a time-independent matrix, $B_s = B$, the Lappo-Danilevskiî condition $B_s \int_t^s B_v dv = \int_t^s B_v dv B_s$ always holds. By Lemma 4.2.1 in Adrianova (1995), the solution to (6.8) is then given as

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}^{i+1}}[G_t] &= \exp\left(\int_0^t B_s ds\right) X_0 = \exp(t \cdot B)\, X_0 \\ &= \exp\Big(t \cdot \left[\mathcal{A}' + \text{diag}(\zeta_{N-1}(e_1, u), \dots, \zeta_{N-1}(e_M, u))\right]\Big) X_0.\end{aligned}$$

Set $C_{N-1}(u) = \mathcal{A}' + \text{diag}(\zeta_{N-1}(e_1, u), \dots, \zeta_{N-1}(e_M, u))$. As $1_M := (1, \dots, 1)' \in \mathbb{R}^M$ and X is taking its values in $E = \{e_1, \dots, e_M\}$, we have $\langle 1_M, X_t \rangle = 1$. Hence,

$$\Phi_{N-1}(u, t) = \Big\langle 1_M, \exp(C_{N-1}(u) \cdot t)\, X_0 \Big\rangle.$$

□

Remark 6.3 (Limitations and Possible Extensions)**.** *Note that a most convenient simplification of* $\Phi_{N-1}(u,t)$ *in the form*

$$\begin{aligned}\Phi_{N-1}(u,t) &= \Big\langle 1_M, \exp\left(\mathcal{A}'\cdot t\right)\cdot\exp\left(\mathit{diag}\left(\zeta_{N-1}(e_1,u),\ldots,\zeta_{N-1}(e_M,u)\right)\cdot t\right)X_0\Big\rangle\\ &= \Big\langle 1_M, C\cdot \mathit{diag}\left(e^{\lambda_1\cdot t},\ldots,e^{\lambda_M\cdot t}\right)\cdot C^{-1}\\ &\quad\times \mathit{diag}\left(e^{\zeta_{N-1}(e_1,u)\cdot t},\ldots,e^{\zeta_{N-1}(e_M,u)\cdot t}\right)X_0\Big\rangle,\end{aligned}$$

as proposed in Elliott and Wilson (2003), with $\lambda_1,\ldots,\lambda_M$ *the real eigenvalues of* $\mathcal{A}$*, and* C *the matrix consisting of the corresponding eigenvectors* $\{c_1,\ldots,c_M\}$*, is in general not possible. This follows from the fact that for matrix exponentials,* $\exp(A+B)=\exp(A)\exp(B)$ *usually does not hold. The second bad news is that whenever the coefficient functions are not constant in time, i.e.*

$$\sigma_i(t,X_{t-})\not\equiv\sigma_i(X_{t-})\quad \text{and}\quad \psi_i(t,X_{t-},z)\not\equiv\psi_i(X_{t-},z),$$

the Lappo-Danilevskiî condition is usually violated and a solution cannot be derived so easily.

Note, however, that there are techniques how the linear homogeneous system (6.8) *may be approximated numerically in the case that the coefficient functions are not state-dependent constant and* ζ_{N-1} *depending on* s*. One possible way is the so-called* Magnus expansion *(going back to Magnus (1954)) yielding the characteristic function to be given as*

$$\phi_{N-1}(u,t)=\Big\langle 1_M,\Big[\prod_{k=0}^{n_{T_{N-1}}}\exp\left(\Omega\left(t_k,t_{k-1}\right)\right)\Big]\cdot X_0\Big\rangle,$$

on an approximation grid covering the whole interval $[0,T_{N-1}]$*, and all* $\Omega(t_k,t_{k-1})$ *are linear combinations of integrals and nested commutators involving the matrix* B_s *on the corresponding interval* $(t_{k-1},t_k]$*. Further details on the employment of the Magnus expansion can be found in Blanes et al. (2009).*

6.2. Determining the Characteristic Function of $Y_j,\ j = 1, \ldots, N-2$

Unlike for the terminal LIBOR rate L_{N-1}, the compensator measures for all other LIBOR rates are non-deterministic. Consequently, the proof of Proposition 6.2 cannot be straight-forwardly extended to the pricing of caplets on other LIBOR rates. Notwithstanding, using observation (4.15) in Subsection 4.3, the respective compensators $\nu^{i+1}_{X_{t-}}$ can be approximated in terms of the terminal compensator $\nu^{N}_{X_{t-}}$. Writing $\widetilde{L}_i$ for the approximated LIBOR dynamics under (4.15), one observes that in a situation where ψ_i is constant conditional on the state X_{t-}, $\psi_i\left(t, X_{t-}, z\right) = \psi_i\left(X_{t-}, z\right)$, the compensator is state-dependent deterministic with regime-dependent constant jump intensity. This yields the same setting as in Proposition 6.2 and the approximative characteristic function of $\widetilde{Y}_i = \ln \widetilde{L}_i\left(t\right) - \ln \widetilde{L}_i\left(0\right)$, $i = 1, \ldots, N-2$ can be determined accordingly.

6.3. Swaption Pricing

Just like in the caplet case (6.2), the price of a swaption can be written as

$$\text{Swaption}_{\alpha,\beta}\left(0\right) = C_{\alpha,\beta}\left(0\right) \cdot K \cdot \mathbb{E}_{\alpha,\beta}\left[\left(e^{Y_{\alpha,\beta}(T_i) - k_{\alpha,\beta}} - 1\right)^+\right], \tag{6.9}$$

where $Y_{\alpha,\beta}\left(t\right) = \ln\left(S_{\alpha,\beta}\left(t\right)/S_{\alpha,\beta}\left(0\right)\right)$ and $k_{\alpha,\beta} = \ln\left(K/S_{\alpha,\beta}\left(0\right)\right)$. Consequently, the price for a swaption is given as

$$\text{Swaption}_{\alpha,\beta}\left(0\right) = K e^{x k_{\alpha,\beta}} C_{\alpha,\beta}\left(0\right) \frac{1}{\pi} \int_0^\infty \text{Re}\left(e^{-iuk_{\alpha,\beta}} \frac{\phi_{\alpha,\beta}\left(ix - u, T_\alpha\right)}{x^2 + x - u^2 + iu\left(1 + 2x\right)}\right) du,$$

for some $x < -1$ and $\phi_{\alpha,\beta}\left(u, t\right)$ the characteristic function of $Y_{\alpha,\beta}$. In order to derive the characteristic function $\phi_{\alpha,\beta}\left(u, t\right)$ of $Y_{\alpha,\beta}$, we go back to the approximation $\widetilde{Y}_{\alpha,\beta}\left(t\right)$ derived in (5.6). By employment of the Doléans-Dade exponential (2.4), the dynamics of $\widetilde{Y}_{\alpha,\beta}\left(t\right)$ are given as

$$\begin{aligned} d\widetilde{Y}_{\alpha,\beta}\left(t\right) = &-\frac{1}{2}\sum_{i,j=\alpha}^{\beta-1} x_i\left(0\right) x_j\left(0\right) \sigma_i\left(t, X_{t-}\right) \sigma_j\left(t, X_{t-}\right) dt + \sum_{j=\alpha}^{\beta-1} x_j\left(0\right) \sigma_j\left(t, X_{t-}\right) dW^N\left(t\right) \\ &- \int_{\mathbb{R}} \widetilde{\psi}_{\alpha,\beta}\left(t, z\right) \nu^{\alpha,\beta}_{X_{t-}}\left(dt, dz\right) + \int_{\mathbb{R}} \ln\left(1 + \widetilde{\psi}_{\alpha,\beta}\left(t, X_{t-}, z\right)\right) \mu\left(dt, dz\right). \end{aligned} \tag{6.10}$$

We furthermore approximate the compensator $\nu^{\alpha,\beta}_{X_{t-}}$ as in (5.1) and (5.2). The adoption of Proposition

6.2 to the pricing of swaption then reads as follows:

Corollary 6.4 (Approximative Characteristic Function of $Y_{\alpha,\beta}$).
Given the same conditions as in Proposition 6.2, the characteristic function $\widetilde{\phi}_{\alpha,\beta}(u,t) = \mathbb{E}_{\mathcal{Q}^{\alpha,\beta}}\left[\exp\left(iu\widetilde{Y}_{\alpha,\beta}(t)\right)\right]$ *of* $\widetilde{Y}_{\alpha,\beta}$ *is given as*

$$\widetilde{\phi}_{\alpha,\beta}(u,t) = \langle 1_M, \exp\left(\widetilde{\mathcal{C}}_{\alpha,\beta} \cdot t\right) X_0 \rangle,$$

where $\widetilde{\mathcal{C}}_{\alpha,\beta}$ *is given as* $\widetilde{\mathcal{C}}_{\alpha,\beta} = \mathcal{A}' + diag\left(\widetilde{\zeta}_{\alpha,\beta}(e_1,u), \ldots, \widetilde{\zeta}_{\alpha,\beta}(e_M,u)\right)$, *and*

$$\begin{aligned}\widetilde{\zeta}_{\alpha,\beta}(e_k,u) \coloneqq & -\frac{u^2}{2}\sum_{i,j=\alpha}^{\beta-1} x_i(0)\, x_j(0)\, \sigma_i(e_k)\, \sigma_j(e_k) - \frac{1}{2} iu \sum_{i,j=\alpha}^{\beta-1} x_i(0)\, x_j(0)\, \sigma_i(e_k)\, \sigma_j(e_k) \\ & - iu\widetilde{\lambda}^{\alpha,\beta}(e_k)\int_{\mathbb{R}} \widetilde{\psi}_{\alpha,\beta}(e_k,z)\, \widetilde{k}^{\alpha,\beta}(e_k,dz) \\ & + \widetilde{\lambda}^{\alpha,\beta}(e_k,dz)\int_{\mathbb{R}}\left[\exp\left(iu\ln\left(\widetilde{\psi}_{\alpha,\beta}(e_k,z)+1\right)\right)-1\right]\widetilde{k}^{\alpha,\beta}(e_k,dz).\end{aligned}$$

7. Calibration

The ultimate step is to investigate the extent in which the MSJD model is suited to reproduce the particular market features observed in the interest rate markets. In order to so, we develop a possible calibration procedure that is suitable for finding parameters that can accurately reproduce the observed market prices. Here, we only investigate the calibration to caplets/caps, but an application to the pricing of swaptions is just as possible. With the LIBOR rate dynamics being given as

$$\frac{dL_i(t)}{L_i(t-)} = \sigma_i(t,X_{t-})\, dW^{i+1}(t) + \int_{\mathbb{R}} \psi_i(t,X_{t-},z)\left(\mu - \nu^{i+1}_{X_{t-}}\right)(dt,dz), \tag{7.1}$$

the fitting procedure entails determining the parameters specifying the rate matrix $\mathcal{A}$, the volatilities $\sigma_i(t,X_{t-})$, the jump function $\psi_i(t,X_{t-},z)$ and the compensator $\nu^{i+1}_{X_{t-}}(dt,dz)$, $i=1,\ldots,N-1$, for a given observation period of market data. For the sake of simplicity, the following assumptions with respect to (7.1) are made:

- The Markov chain takes its values in a state space with only two states, $E = \{e_1, e_2\}$.

- Wiener processes and mark space are one-dimensional.
- Volatilities $\sigma_i(t, X_{t-}) \equiv \sigma_i(X_{t-})$ and jump terms $\psi_i(t, X_{t-}, z) = e^z - 1$ are regime-dependent constant.
- The marker distribution of the $\mathcal{Q}^N$-compensator $\nu^N_{X_{t-}}$ is Markov-switching Gaussian,

$$\nu^N_{X_{t-}}(dt, dz) = \lambda^N(X_{t-}) \frac{1}{\sqrt{2\pi v_J^2(X_{t-})}} \exp\left(-\frac{(z - m_J(X_{t-}))^2}{2v_J^2(X_{t-})}\right) dz\, dt.$$

- The $\mathcal{Q}^{i+1}$-compensators $\nu^{i+1}_{X_{t-}}$, $1 \le i \le N-2$, are approximated as

$$\widetilde{\nu}^{i+1}_{X_{t-}}(dt, dz) = \prod_{j=i+1}^{N-1} \left(1 + \frac{\delta L_j(0)(e^z - 1)}{1 + \delta L_j(0)}\right) \nu^N_{X_{t-}}(dt, dz). \tag{7.2}$$

7.1. The Data

The data used in the calibration of the MSJD extension of the LMM encompassed a time period of more than 9 years, with all trading days between 2003/01/01 and 2012/06/22 being considered (2464 days). The data source was Thomson Reuters. As the provided discount curves occasionally displayed unrealistic edges and humps, as well as non-decreasing behavior for increasing maturities, they were interpolated for all days by a cubic spline through the bond prices corresponding to maturities 3m, 6m, 9m, 1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, and 10y. Bond prices/discount factors for different maturities were then derived from the interpolated curves. Investigated cap prices were based on 3m LIBOR rates, which were in turn derived from the discount curve. In detail, we considered USD ATM caps with maturities 1y, 2y, 3y, 4y, 5y, 7y and 10y years, quoted in volatilities v^1, v^2, v^3, v^4, v^5, v^7 and v^{10}. Caplet prices were derived from the available cap information using a bootstrapping technique. For the caps with given maturities, prices for 39 caplets with maturities $(0.25, \ldots, 9.75)$ and expiries $(0.5, \ldots, 10)$ were deduced. Figure 2 depicts the (forward) volatility of the caplets with maturities 2y, 2.25y, 2.5y and 2.75y.[5]

[5] An instructive example for employment of the bootstrapping technique can be found in Filipovic (2009), p. 215. Observe that the bootstrapping procedure assumes constant volatilities for caplets in between caps of succeeding maturity. For example, for $t = 0$, the (identical) volatilities $\sigma_{1\mathrm{y}}$ for caplets with maturities 1y, 1.25y, 1.5y and 1.75y are derived based on the

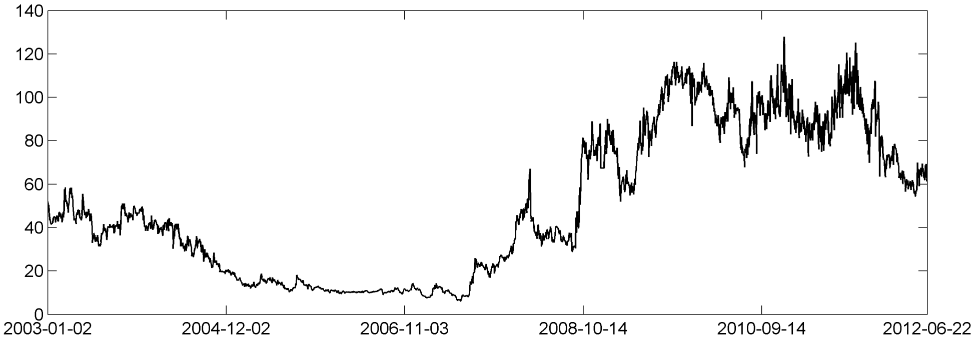

Figure 2: Fitted (forward) caplet volatility (in %) for maturities 2y, 2.25y, 2.5y and 2.75y.

7.2. Determining the States of the Markov Chain

In the first step of the calibration routine, the respective state of the underlying Markov chain X at each day of the observation period needed to be identified. The market-implied cap volatilities were chosen as an appropriate indicator for the overall market movement and hereby the evolvement of X. In order to find both, the infinitesimal generator $\mathcal{A}$ and the most likely state of X at each time point of the observation period, the seven different time series of cap volatilities v^k were considered separately. With the quantities of interest being volatilities, it seemed sensible to model each v^k as a Markov-switching Vasicek process

$$dv^k(t) = \kappa^k(X_t)\big(\theta^k(X_t) - v^k(t)\big)dt + s^k(X_t)\,dW^{\mathbb{P}}(t)$$

and to employ an Bayesian inference algorithm to infer the most likely path, the infinitesimal generator of the underlying Markov chain as well as the regime-dependent parameters for each of the respective processes. To this end, the Markov-switching Vasicek processes were approximated as Markov-switching AR(1) processes (compare, e.g., Gray (2002)). The respective results for the different caps were averaged and rounded to either 0 or 1 to receive an overall most likely path of the Markov chain. In a final step, the infinitesimal generator $\mathcal{A}$ was derived by running another MCMC algorithm for the previously derived most likely path of the Markov chain. The stationary distribution $\pi = (\pi_1, \pi_2)'$ of $\mathcal{A}$

formula

$$\begin{aligned}\text{Cap}(0, 2\text{y}, K_{2\text{y}}) - \text{Cap}(0, 1\text{y}, K_{1\text{y}}) =& \text{Caplet}(0, 1\text{y}, \sigma_{1\text{y}}, K_{2\text{y}}) + \text{Caplet}(0, 1.25\text{y}, \sigma_{1\text{y}}, K_{2\text{y}}) \\ &+ \text{Caplet}(0, 1.5\text{y}, \sigma_{1\text{y}}, K_{2\text{y}}) + \text{Caplet}(0, 1.75\text{y}, \sigma_{1\text{y}}, K_{2\text{y}})\ ,\end{aligned}$$

where $\text{Caplet}(0, T_j, \sigma_{1\text{y}}, K_{2\text{y}})$ denotes the price of a caplet at $t = 0$ with maturity T_j, expiry T_{j+1} and ATM cap strike $K_{2\text{y}}$ for maturity 2y, evaluated by the Black (1976)-formula for constant volatility $\sigma_{1\text{y}}$.

was determined via the relation $\pi'\mathcal{A} = \mathbf{0}$, with $\pi_1 + \pi_2 = 1$.

Note that the derived infinitesimal generator $\mathcal{A}$ of X is in fact the generator with respect to the physical measure $\mathbb{P}$. However, in accordance with usual practice, we take $\mathcal{A}$ to be the already risk-adjusted generator under $\mathcal{Q}$ (compare Maul (2012)). By Remark 4.4, $\mathcal{A}$ is then also the infinitesimal generator of X with respect to the forward measures $\mathcal{Q}^2$,...,$\mathcal{Q}^{40}$.

7.3. Fitting the Volatility and Jump Parameters

Next, the time series was separated according to the regimes determined in Section 7.2. As the pricing formula (6.3) is rather involved, the calibration procedure for volatility and jump parameters was further split up: First the parameters of the LIBOR rate with longest maturity, L_{N-1}, were estimated on a step-by-step basis. These comprise $\sigma_{N-1}(e_j)$, $\lambda^N(e_j)$, $m_J(e_j)$, and $v_J^2(e_j)$, $j = 1, 2$. Through relation (7.2), the knowledge about the terminal compensator $\nu^N_{X_{t-}}$ then allowed us to approximate the compensators of all other forward measures. Consequently, the second step of the fitting routine simply entailed determining the volatilities $\sigma_1(e_j), \ldots, \sigma_{N-2}(e_j)$, $j = 1, 2$, for the remaining LIBOR rates.

Determining the Parameters Related to L_{N-1}

A straight-forward approach to fitting the parameters for the LIBOR rate L_{N-1} would be a least-square minimization over some subperiod of the observation time period.[6] To this end, we would consider the sum of least squares,

$$\sum_{l=1}^{k} \frac{1}{\left(\mathrm{Caplet}^{\mathrm{MKT}}_{N-1}(t_l)\right)^2} \left(\mathrm{Caplet}^{\mathrm{MOD}}_{N-1}(t_l) - \mathrm{Caplet}^{\mathrm{MKT}}_{N-1}(t_l)\right)^2, \tag{7.3}$$

where k is some larger number, and $t_1, \ldots, t_k$ are some successive time points, $\mathrm{Caplet}^{\mathrm{MKT}}_{N-1}$ is the market-observed price and $\mathrm{Caplet}^{\mathrm{MOD}}_{N-1}$ is the model price determined by the pricing formula (6.3). The prefactors ensure that all summands are of about the same scale. Surprisingly at first sight, (7.3) constitutes an under-determined problem. To understand why this is the case, recall that the main role in the pricing formula (6.3) is taken by the characteristic function. Following Proposition 6.2, its evaluation basically

[6] An example for non-Markov-switching jump diffusion illustrating this approach can, e.g., be found in Cont and Tankov (2004).

reduces to an eigenvalue problem, i.e., the calculation of the matrix exponential

$$\begin{aligned}\exp\left(C_{N-1}(u)\right) &= \exp\left(t\cdot\begin{bmatrix}-a_{1,2}+\zeta_{N-1}(e_1,u) & a_{2,1}\\ a_{1,2} & -a_{2,1}+\zeta_{N-1}(e_2,u)\end{bmatrix}\right)\\ &= D\cdot\mathrm{diag}(e^{\lambda_1},e^{\lambda_2})\cdot D^{-1},\end{aligned}$$

where λ_1,λ_2 are the eigenvalues of $C_{N-1}(u)$ and D is the matrix of the corresponding eigenvectors. As commonly known, the eigenvalues are the roots of the characteristic polynomial,

$$\begin{aligned}\lambda_1,\lambda_2 = &-\frac{1}{2}\left(a_{1,2}-\zeta_{N-1}(e_1,u)+a_{2,1}-\zeta_{N-1}(e_2,u)\right)\\ &\pm\sqrt{\frac{1}{4}\left(a_{1,2}-\zeta_{N-1}(e_1,u)+a_{2,1}-\zeta_{N-1}(e_2,u)\right)^2-a_{1,2}a_{2,1}}.\end{aligned}\tag{7.4}$$

Two things become apparent from representation (7.4): First, $\zeta_{N-1}(e_1,u)$ and $\zeta_{N-1}(e_2,u)$ are completely interchangeable, and it is hence impossible to determine whether $\zeta_{N-1}(e_1,u)$ or $\zeta_{N-1}(e_2,u)$ takes a bigger value. Even worse, there might be infinitely many pairs of $(\zeta_{N-1}(e_1,u),\zeta_{N-1}(e_2,u))$ that satisfy relation (7.4), for a given pair of eigenvalues, and we have an under-determined system (as there are two variables for one equation). As a second problem, it is impossible to determine with what weight the volatility part and the jump part of ζ_{N-1} enter into the characteristic function. It would be even possible to completely disregard either one of the parts and yet be able to fit the market prices just as well as if both parts played a role. As a result of these observations, there are infinitely many parameters that will yield a minimal squared distance between model and market prices.

To avoid the problem, we pursued a different approach than a least-squares minimization:

(i) Based on the market-observed LIBOR rate movements for L_{N-1}, we determined an estimate for the jump intensity $\lambda^{\mathbb{P}}(e_j)$ and the jump size distribution $k^{\mathbb{P}}(e_j,\cdot)$, $j=1,2$ by considering the logarithmized and discretized dynamics of $\widetilde{Y}_{N-1}(t):=\ln(L_{N-1}(t)/L_{N-1}(t-1))$,

$$\widetilde{Y}_{N-1}(t)=\hat{\alpha}_{N-1}+\sigma_{N-1}(W^{\mathbb{P}}(t_j)-W^{\mathbb{P}}(t_{j-1}))+\xi^{\mathbb{P}}_{N-1}(t_j,X_{t_j})J^{\mathbb{P}}_{N-1}(t_j,X_{t_j}).$$

with $\xi^{\mathbb{P}}_{N-1}$ the jump size and $J^{\mathbb{P}}_{N-1}$ a discrete random variable taking either value 0 (“no jump”) or

1 ("jump") in each interval. For details, see, e.g., Johannes et al. (1999). The estimation is based on an MCMC algorithm for jump diffusions. Observe that the derived parameters are $\mathbb{P}$-quantities.

(ii) For every day of each time series corresponding to state e_1 and e_2, we used the market-observed caplet prices Caplet_{N-1} and ran a nonlinear optimization routine to derive a first estimate of the volatility, $\widetilde{\sigma}_{N-1}(e_j)$, $j = 1, 2$. This estimate was retrieved by ignoring the jump part in the pricing formula. In order to guarantee for a unique solution, we let

$$\mathcal{Q}^N(X_t = e_1) \cdot \widetilde{\sigma}_{N-1}(e_1) + \mathcal{Q}^N(X_t = e_2) \cdot \widetilde{\sigma}_{N-1}(e_2) = \sigma_{N-1}^{\text{MKT}}, \tag{7.5}$$

which reflects the reasonable assumption that the market-observed volatility is the investors' expected volatility, based on two possible states for each day with corresponding volatilities $\widetilde{\sigma}_{N-1}(e_1)$ and $\widetilde{\sigma}_{N-1}(e_2)$.

(iii) The inaccuracy in overestimating volatility in Step (ii) was corrected in the final step where, based on the MSJD dynamics (7.1), the results from Step (i) and (ii) were combined. Again given daily market prices Caplet_{N-1}, another nonlinear optimization routine was run to determine the $\mathcal{Q}^N$-parameters σ_{N-1}, λ^N, m_J^N and v_J^N. Solutions were found by postulating further requirements: The proportionality between $\widetilde{\sigma}_{N-1}(e_1)$ and $\widetilde{\sigma}_{N-1}(e_2)$ is conserved by requiring

$$\frac{\sigma_{N-1}(e_1)}{\sigma_{N-1}(e_2)} = \frac{\widetilde{\sigma}_{N-1}(e_1)}{\widetilde{\sigma}_{N-1}(e_2)}.$$

For the jump parameters, it is assumed that the jump size distribution is not influenced by the measure change from $\mathbb{P}$ to $\mathcal{Q}^N$, $k^N(X_{t-}, \cdot) \equiv k^{\mathbb{P}}(X_{t-}, \cdot)$. In contrast, the jump intensity differs between $\mathbb{P}$ and $\mathcal{Q}^N$. Assuming that the intensities do not change over time, by Girsanov's Theorem 2.3 there exists a $\phi(X_{t-}) \equiv c$ such that $\left(\lambda^N(e_1), \lambda^N(e_2)\right) = c \cdot \left(\lambda^{\mathbb{P}}(e_1), \lambda^{\mathbb{P}}(e_2)\right)$. Last, but not least, we postulate that

$$\begin{aligned} 120\% \cdot (\text{integral evaluated without jump part}) &\leq \text{integral evaluated with jump part} \\ &\leq 200\% \cdot (\text{integral evaluated without jump part}). \end{aligned} \tag{7.6}$$

This seems to be a reasonable assumption as both, volatility and jump, should have a significant influence on the evaluation of the characteristic function.

Note that only every tenth day in the available time series is considered, a choice in favor of computation time. Nonetheless, there should be no worries that this limitation could be of data-distorting character, as the considered time points were chosen to be equidistant and no further limiting assumptions were made.

Determining the Parameters Related to $L_1, \dots, L_{N-2}$

As the jump part in the dynamics of every LIBOR rate had already been completely determined by the results of the previous subsection, the only step that remained to be done was the derivation of volatilities $\sigma_i(e_j)$, $j = 1, \dots, N-2$, $j = 1, 2$. The fitting was done on a daily basis, by again first fitting a model without jumps to find estimates $(\widetilde{\sigma}_i(e_1), \widetilde{\sigma}_i(e_2))$ and then reintroducing jumps into the model. Again, the actual volatilities $(\sigma_i(e_1), \sigma_i(e_2))$ were set to have the the proportion as the first estimates $(\widetilde{\sigma}_i(e_1), \widetilde{\sigma}_i(e_2))$, yielding a unique solution of the fitting problem. The jump parameters are set according to the calibration results from the previous section. Again, the `fmincon` algorithm was used for fitting.

7.4. Discussion of the Results of the Calibration

All market-observed caplet prices could be perfectly reproduced within the MSJD extension. Both Gibbs samplers, for the determination of the parameters describing the Markov chain and the jump parameters, displayed very good sampling behavior. For both samplers, the generated Markov chain mixed well, as the parameter space was exploited nicely, and after a very short burn-in, convergence was reached. It may hence be concluded that this extension is a suitable generalization to existing LIBOR market models and can help to account for jumps and overall market trends. The detailed results for the different steps of the calibration are as follows:

For the most likely state of the Markov chain at different time points, the path estimates for the different cap volatilities are very similar. Figure 3 displays the result for the time series of a cap with maturity 1y. This gives the ex post justification of considering all cap volatilities independently and then taking the average path. Based on this average path, the rate matrix $\mathcal{A}$ and the corresponding stationary distribution

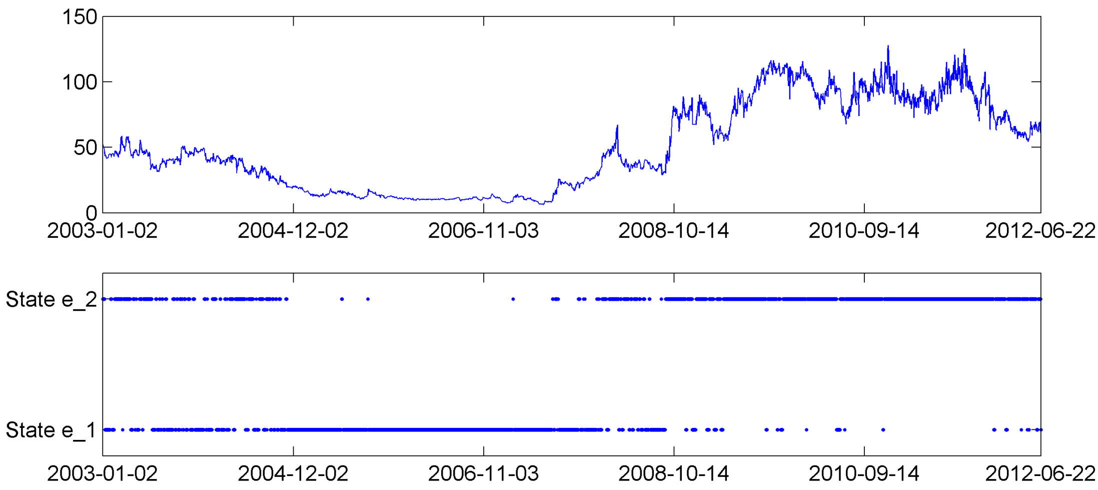

Figure 3: Estimated states of the economy at all days of the time series, for cap volatilities v^1.

π are given as

$$\mathcal{A} = \begin{pmatrix} -10.7910 & 10.7910 \\ 17.9111 & -17.9111 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} 0.6239 \\ 0.3761 \end{pmatrix}.$$

The findings coincide with the intuition that the market is more likely to reside in a ‘normal’ state than in the excited state as $\pi_1 \gg \pi_2$ (see also Svoboda (2005)).

For Step (i), the sampled parameters in Table 1 display that there is a tendency in state e_1 towards jumps into a positive direction, while jumps in state e_2 have a slightly downwards tendency. This coincides with the intuition that in times of crises and market unrest, a downwards tendency should also manifest itself in the jump sizes. Also, as $\lambda^{\mathbb{P}}(e_2) > \lambda^{\mathbb{P}}(e_1)$, it seems that in times of crises, jumps happen more often.

For Step (ii), the caplet prices were perfectly recovered at all times. It turned out that $(\widetilde{\sigma}_{N-1}(e_1), \widetilde{\sigma}_{N-1}(e_2))$ and $\sigma^{\text{MKT}}_{N-1}$ are in fact fairly close together for all time points. Finally, for Step (iii), we find that an estimate of $c = 0.6$ yields very good results for the algorithm minimizing the distance between model and market prices. Only for few points, the constraints do not allow to perfectly reproduce

Parameters	Sampled Values	Standard Error
$(\lambda^{\mathbb{P}}(e_1), \lambda^{\mathbb{P}}(e_2))$	$(0.1823, 0.2263)$	$(8.6105 \cdot 10^{-05}, 3.1956 \cdot 10^{-04})$
$(m_J(e_1), m_J(e_2))$	$(0.0014, -0.0053)$	$(1.2693 \cdot 10^{-05}, 1.8959 \cdot 10^{-05})$
$(v_J^2(e_1), v_J^2(e_2))$	$(0.0026, 0.0026)$	$(1.2317 \cdot 10^{-06}, 2.8182 \cdot 10^{-06})$

Table 1: Sampled jump parameters from the MCMC algorithm for jump diffusions.

the market price through the model. It follows that $\left(\lambda^N(e_1), \lambda^N(e_2)\right) = 0.6 \cdot \left(\lambda^{\mathbb{P}}(e_1), \lambda^{\mathbb{P}}(e_2)\right) = (0.1091, 0.1391)$, yielding a similar result to the calibration for the (non Markov-switching) jump diffusion model of Belomestny and Schoenmakers (2011) where the jump intensity is estimated as 0.1.

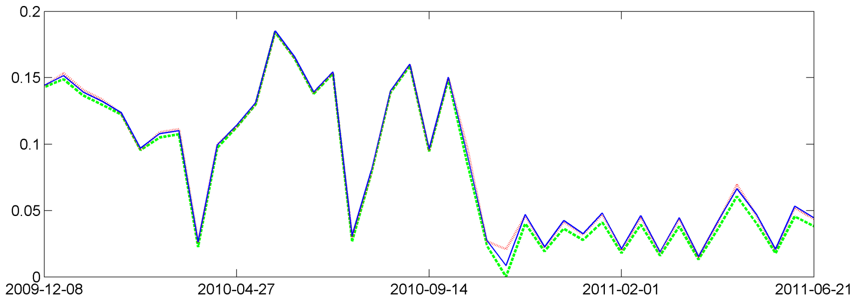

Figure 4: Fitting of $\sigma_{9.75}(e_1)$ (dashed) and $\sigma_{9.75}(e_2)$ (dotted) and the observed market volatility (solid) for the caplet with longest maturity 9.75 (in %). The depicted time frame is 2009-12-08 until 2011-06-21.

For fixed $c = 0.6$, the results for the true volatilities $(\sigma_{N-1}(e_1), \sigma_{N-1}(e_2))$ are depicted in Figure 4. All market prices in the investigated observation period were perfectly reproduced. An ex post investigation of the limits set in (7.6) indicate that the limits were always at least nearly respected. Last, Figure 5 shows the fitting of the volatilities for the LIBOR rate with maturity 3.75. Also here, caplet prices $\text{Caplet}_{3.75}$ were perfectly reproduced for all time points.

8. Conclusion

In this paper, we have addressed some of the problems arising in the context of the log-normal LIBOR market model by replacing the ordinary diffusion of the original model by a Markov-switching jump diffusion process. By doing so, we have encountered both, the need to model different economic phases as well as suddenly occurring jumps, as observed in market data. By exploiting the relation between bond

Figure 5: Fitting of $\sigma_{3.75}(e_1)$ (dashed) and $\sigma_{3.75}(e_2)$ (dotted) and the observed market volatility (solid) for the caplet with maturity 3.75, $c = 0.6$. The depicted time frame is 2003-10-09 until 2005-04-21.

prices, forward rates, and simple rates, we successfully showed that such an extension to the original model can be embedded into a generalized Markov-switching Heath-Jarrow-Morton (HJM) model. We thereby proved that our model extension is free of arbitrage. The LIBOR rates were shown to follow Markov-switching jump diffusions without drift under their respective forward measure. With measure changes playing the central role within the model derivation, we proved the central result that measure changes between forward measures and the risk-neutral measure have no effect on the infinitesimal generator of the underlying Markov chain. Wiener processes and Markov-switching compensator, however, *do* follow different dynamics under different measures and we derived expressions for their interrelations. While the introduced interest rate dynamics are rich enough to incorporate both sudden shocks and structural market movement into the model, we showed that they are still simple enough to allow for the pricing of caps/floors. Under the assumption of volatilities being modeled as Markov-switching constants, we showed that the characteristic functions of the logarithmized LIBOR rates needed for pricing with Laplace transforms can be derived in analytical form. Eventually, we demonstrated in the last section that the model can be successfully calibrated to market-observed cap/caplet prices, even though this procedure is related to a considerable amount of effort. In particular, it is necessary to make additional reasonable assumptions to further specify the parameters to be estimated. Ex post, all assumptions made turned out to be reasonable, as market prices could be perfectly recovered in all cases considered. In summary, the MSJD extension to the LIBOR Market Model has been shown to not only considerably enrich the possible dynamics under which interest rates can be modeled, but also provides an appropriate basis for further research and discussion.

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A. Proof of Proposition 4.1

The proof for 1. follows along the lines of reasoning in Brace et al. (1997). The $\mathcal{Q}$–dynamics of L_i, $i = 1, \ldots, N-1$, are determined by deriving (4.6) on both sides and using the no-arbitrage condition (4.4). In detail, let $\tau = T - t$, $\tau- = T - t-$ and $\tau_i = T_i - t$. Also, set $r(t,\tau) = f(t, t+\tau) = f(t,T)$ and $K_i(t,\tau_i) := L_i(t, t+\tau_i, t+\tau_i+\delta)$. Let

$$a^*(t, X_{t-}, \tau) = \alpha^*(t, X_{t-}, T), \quad s^*(t, X_{t-}, \tau) = \varsigma^*(t, X_{t-}, T), \\ g^*(t, X_{t-}, \tau, z) = \gamma^*(t, X_{t-}, T, z) \tag{A.1}$$

and

$$a(t, X_{t-}, \tau_i) = \alpha_i(t, X_{t-}), \quad s(t, X_{t-}, \tau_i) = \varsigma_i(t, X_{t-}), \quad g(t, X_{t-}, \tau_i, z) = \gamma_i(t, X_{t-}, z). \tag{A.2}$$

Using $\frac{\partial}{\partial \tau} r(t,\tau) = \frac{\partial}{\partial T} f(t,T)$, it follows from (4.3) that the dynamics of r are given as

$$\begin{aligned} dr(t,\tau) &= df(t, t+\tau) + \frac{\partial}{\partial T} f(t, t+\tau)\, dt \\ &= a^*(t, X_{t-}, \tau)\, dt + s^*(t, X_{t-}, \tau)\, dW^{\mathcal{Q}}(t) \\ &\quad + \int g^*(t, X_{t-}, \tau, z)(t,\tau,z)\left(\mu(dt,dz) - \nu^{\mathcal{Q}}_{X_{t-}}\right)(dt,dz) + \frac{\partial}{\partial \tau} r(t,\tau)\, dt. \end{aligned} \tag{A.3}$$

By (4.5) and (A.1),

$$\begin{aligned} a^*(t, X_{t-}, \tau) &= -\frac{\partial}{\partial \tau} a(t, X_{t-}, \tau), \quad s^*(t, X_{t-}, \tau) = -\frac{\partial}{\partial \tau} s(t, X_{t-}, \tau), \\ g^*(t, X_{t-}, \tau, z) &= -\frac{\partial}{\partial \tau} g(t, X_{t-}, \tau, z). \end{aligned}$$

Inserting this into (A.3) yields

$$\begin{aligned} dr(t,\tau) = \frac{\partial}{\partial \tau}\Big[&(r(t,\tau) - a(t, X_{t-}, \tau))\, dt \\ &- s(t, X_{t-}, \tau)\, dW^{\mathcal{Q}}(t) - \int_{\mathbb{R}^k} g(t, X_{t-}, \tau, z)\left(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\right)(dt,dz)\Big]. \end{aligned} \tag{A.4}$$

A. Proof of Proposition 4.1

Let

$$X^i(t) := \ln\left(\frac{B(t, t+\tau_i)}{B(t, t+\tau_i+\delta)}\right) = \int_{t+\tau_i}^{t+\tau_i+\delta} f(t,u)\,du = \int_{\tau_i}^{\tau_i+\delta} r(t,u)\,du.$$

The derivation of X^i in component t yields, due to (A.4),

$$\begin{aligned} dX^i(t) = {} & \left(r(t,\tau_{i+1}) - r(t,\tau_i) + a(t, X_{t-}, \tau_i) - a(t, X_{t-}, \tau_{i+1})\right) dt \\ & + \left(s(t, X_{t-}, \tau_i) - s(t, X_{t-}, \tau_{i+1})\right) dW^{\mathcal{Q}}(t) \\ & + \int_{\mathbb{R}^k} \left(g(t, X_{t-}, \tau_i, z) - g(t, X_{t-}, \tau_{i+1}, z)\right) \left(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\right)(dt, dz). \end{aligned}$$

The application of Itō's Lemma 2.1 to $K_i(t,\tau_i) = \delta^{-1}\left[\exp(X^i(t)) - 1\right]$, furthermore gives, after some rearrangements,

$$\begin{aligned} dK_i(t,\tau_i) = \exp(X^i(t)) \Bigg[\Big[& r(t,\tau_{i+1}) - r(t,\tau_i) + a(t, X_{t-}, \tau_i) - a(t, X_{t-}, \tau_{i+1}) \\ & + \frac{1}{2} \left\| s(t, X_{t-}, \tau_i) - s(t, X_{t-}, \tau_{i+1}) \right\|^2 \Big] dt \\ & + \left[s(t, X_{t-}, \tau_i) - s(t, X_{t-}, \tau_{i+1})\right]' dW^{\mathcal{Q}}(t) \\ & + \int_{\mathbb{R}^k} \Big(g(t, X_{t-}, \tau_i, z) - g(t, X_{t-}, \tau_{i+1}, z) \left(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\right)(dt, dz) \Big) \\ & + \int_{\mathbb{R}^k} \Big(e^{g(t, X_{t-}, \tau_i, z) - g(t, X_{t-}, \tau_{i+1}, z)} - 1 \\ & \qquad - \left(g(t, X_{t-}, \tau_i, z) - g(t, X_{t-}, \tau_{i+1}, z)\right) \Big) \mu(dt, dz) \Bigg]. \end{aligned}$$

Because $K_i(t,\tau_i) = \delta^{-1}\left[\exp(X^i(t)) - 1\right]$, it follows that

$$dK_i(t,\tau_i) = \frac{1}{\delta}\left(1 + \delta K_i(t-, \tau_i-)\right)\left[\ldots\right]. \tag{A.5}$$

Observe next that $dL_i\,(t) = dK_i\,(t,\tau_i) - \frac{\partial}{\partial \tau_i} K_i\,(t,\tau_i)$. Hence, by (A.2) and (A.5),

$$\begin{aligned} dL_i\,(t) = \frac{1}{\delta}\,(1+\delta L_i\,(t-)) \Bigg[\Bigg[& \alpha_i\,(t,X_{t-}) - \alpha_{i+1}\,(t,X_{t-}) + \frac{1}{2}\left\| \varsigma_i\,(t,X_{t-}) - \varsigma_{i+1}\,(t,X_{t-}) \right\|^2 \Bigg] dt \\ & + \left[\varsigma_i\,(t,X_{t-}) - \varsigma_{i+1}\,(t,X_{t-}) \right]' dW^{\mathcal{Q}}\,(t) \\ & + \int_{\mathbb{R}^k} \Big(\gamma_i\,(t,X_{t-},z) - \gamma_{i+1}\,(t,X_{t-},z) \left(\mu - \nu^{\mathcal{Q}}_{X_{t-}} \right)(dt,dz) \Big) \\ & + \int_{\mathbb{R}^k} \Big(e^{\gamma_i(t,X_{t-},z)-\gamma_{i+1}(t,X_{t-},z)} - 1 - \left(\gamma_i\,(t,X_{t-},z) - \gamma_{i+1}\,(t,X_{t-},z) \right) \Big)\, \mu\,(dt,dz) \Bigg]. \end{aligned}$$

Insertion of the no-arbitrage condition (4.4) immediately yields (4.7).

For 2., the Change-of-Numéraire Technique 2.2 is applied. The Radon-Nikodým derivative for changing from the risk-neutral measure $\mathcal{Q}$ to the forward measure $\mathcal{Q}^{i+1}$ is given by

$$\left. \frac{d\mathcal{Q}^{i+1}}{d\mathcal{Q}} \right|_{\mathcal{F}_t} := M_{i+1}\,(t) := \frac{B\,(t,T_{i+1})}{B_t} \frac{1}{B\,(0,T_{i+1})}.$$

With bonds being the primary traded assets of the market, the processes $(B(t,T_i)/B_t)_{t\in[0,T_i]}$, $i = 1,\ldots,N$, are all $\mathcal{Q}$-martingales. Taking furthermore into account that $B\,(t,T_i) = \exp\left(-\int_t^{T_i} f(t,u)du\right)$, it is rather straightforward (see, e.g., Björk et al. (1997)) that

$$\begin{aligned} d\left(\frac{B(t,T_{i+1})}{B_t}\right) = \left(\frac{B(t-,T_{i+1})}{B_{t-}}\right) \Big[& \varsigma_{i+1}(t,X_{t-})' dW^{\mathcal{Q}}(t) \\ & + \int_{\mathbb{R}^k} \gamma_{i+1}(t,X_{t-},z) \big(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\big)(dt,dz) \Big] \end{aligned}$$

Consequently, the dynamics of M_{i+1} are given as

$$\begin{aligned} dM_{i+1}\,(t) &= \frac{1}{B\,(0,T_{i+1})} d\left(\frac{B\,(t,T_{i+1})}{B_t}\right) \\ &= M_{i+1}\,(t-) \left[\varsigma_{i+1}\,(t,X_{t-})'\, dW^{\mathcal{Q}}\,(t) + \int_{\mathbb{R}^k} \gamma_{i+1}\,(t,X_{t-},z) \left(\mu - \nu^{\mathcal{Q}}_{X_{t-}}\right)(dt,dz) \right]. \end{aligned}$$

By assumption, ς_{i+1} and γ_{i+1} are integrable. In combination with the specification of the jump measure, the conditions of Girsanov's Theorem 2.3 are satisfied, and we may define a $\mathcal{Q}^{i+1}$-Wiener process W^{i+1}

by

$$dW^{i+1}(t) = -\varsigma_{i+1}(t, X_{t-})\, dt + dW^{\mathcal{Q}}(t) \tag{A.6}$$

and a $\mathcal{Q}^{i+1}$-compensator $\nu^{i+1}_{X_{t-}}$ through

$$\nu^{i+1}_{X_{t-}}(dt, dz) = \exp\left(\gamma_{i+1}(t, X_{t-}, z)\right) \nu^{\mathcal{Q}}_{X_{t-}}(dt, dz). \tag{A.7}$$

Inserting (A.6) and (A.7) into dynamics (4.7), the dynamics of $\frac{\delta}{1+\delta L_i(t-)} \cdot dL_i(t)$ are given as

$$\begin{aligned}
\frac{\delta \cdot dL_i(t)}{1 + \delta L_i(t-)} = & \left(\varsigma_i(t, X_{t-}) - \varsigma_{i+1}(t, X_{t-})\right)' dW^{i+1}(t) \\
& + \int_{\mathbb{R}^k} \big(e^{\gamma_i(t,X_{t-},z) - 2\gamma_{i+1}(t,X_{t-},z)} - e^{-\gamma_{i+1}(t,X_{t-},z)} + 1 \\
& \qquad - e^{\gamma_i(t,X_{t-},z) - \gamma_{i+1}(t,X_{t-},z)}\big) \nu^{i+1}_{X_{t-}}(dt, dz) \\
& + \int_{\mathbb{R}^k} \left(e^{\gamma_i(t,X_{t-},z) - \gamma_{i+1}(t,X_{t-},z)} - 1\right) \mu(dt, dz) \\
& + \int_{\mathbb{R}^k} \left(e^{\gamma_i(t,X_{t-},z) - 2\gamma_{i+1}(t,X_{t-},z)} - e^{-\gamma_{i+1}(t,X_{t-},z)}\right) \nu^{i+1}_{X_{t-}}(dt, dz)\,(dt, dz).
\end{aligned}$$

Rearranging the terms yields the claimed dynamics.

For 3., set

$$\varsigma_{i+1}(t, X_{t-}) - \varsigma_i(t, X_{t-}) = \frac{\delta L_i(t)}{1 + \delta L_i(t)} \sigma_i(t, X_{t-}), \tag{A.8}$$

$$e^{\gamma_i(t,X_{t-},z) - \gamma_{i+1}(t,X_{t-},z)} - 1 = \frac{\delta L_i(t)}{1 + \delta L_i(t)} \psi_i(t, X_{t-}, z). \tag{A.9}$$

By substituting (A.8) and (A.9) into (4.8), dynamics (4.9) then immediately follow.

B. Proof of Proposition 5.1

Under the terminal measure $\mathcal{Q}^N$, the dynamics of the k-th LIBOR rate in (4.13) can be written as

$$\frac{dL_k(t)}{L_k(t-)} = \alpha_k(t, X_{t-})\,dt + \sigma_k(t, X_{t-})'\,dW^N(t) + \int_{\mathbb{R}^k} \psi_k(t, X_{t-}, z)\left(\mu - \nu^N_{X_{t-}}\right)(dt, dz),$$

with α_k denoting the drift term

$$\begin{aligned}\alpha_k = &-\sum_{j=k+1}^{N-1} \frac{\delta L_j(t-)}{1+\delta L_j(t-)}\sigma_k(t, X_{t-})'\,\sigma_j(t, X_{t-})\,dt \\ &-\int_{\mathbb{R}^k} \psi_k(t, X_{t-}, z)\left(\prod_{j=k+1}^{N-1}\left(1 + \frac{\delta L_j(t-)}{1+\delta L_j(t)} \cdot \psi_j(t, X_{t-}, z)\right) - 1\right)\nu^N_{X_{t-}}(dt, dz).\end{aligned}$$

The Radon-Nikodým derivative $\eta_{N,\alpha,\beta}$ corresponding to a measure change from $\mathcal{Q}^N$ to $\mathcal{Q}^{\alpha,\beta}$ is by the Change-of-Numéraire Technique 2.2 given as

$$\eta_{N,\alpha,\beta}(t) := \left.\frac{d\mathcal{Q}^{\alpha,\beta}}{d\mathcal{Q}^N}\right|_{\mathcal{F}_t} = \frac{B(0,T_N)}{C_{\alpha,\beta}(0)}\frac{C_{\alpha,\beta}(t)}{B(t,T_N)} = \delta\frac{B(0,T_N)}{C_{\alpha,\beta}(0)}\sum_{i=\alpha}^{\beta-1}\frac{B(t,T_{i+1})}{B(t,T_N)} \tag{B.1}$$

where the definition of the annuity $C_{\alpha,\beta}(t)$ in (3.4) was inserted. For all $\alpha \le i \le \beta - 1$, the term $B(t,T_{i+1})/B(t,T_N)$ is a forward contract on the bond $B(t,T_{i+1})$ with maturity T_N, and therefore a $\mathcal{Q}^N$-martingale. Consequently, $\eta_{N,\alpha,\beta}$ is a $\mathcal{Q}^N$-martingale as well. Inserting

$$\frac{B(t,T_{i+1})}{B(t,T_N)} = \prod_{k=i+1}^{N-1}(1+\delta L_k(t)) \tag{B.2}$$

into (B.1), it follows that

$$d\eta_{N,\alpha,\beta}(t) = \delta\,\frac{B(0,T_N)}{C_{\alpha,\beta}(0)}\left(\sum_{i=\alpha}^{\beta-1} d\left[\prod_{k=i+1}^{N-1}(1+\delta L_k(t))\right]\right). \tag{B.3}$$

By Itō's Lemma 2.1,

$$
\begin{aligned}
& d\Big[\prod_{k=i+1}^{N-1}\left(1+\delta L_k\left(t\right)\right)\Big] \\
& =\left(\ldots\right) dt+\sum_{j=i+1}^{N-1}\frac{\delta L_j\left(t-\right)}{1+\delta L_j\left(t-\right)}\prod_{k=i+1}^{N-1}\left(1+\delta L_k\left(t\right)\right)\Big[\sigma_j\left(t,X_{t-}\right)' dW^N\left(t\right)\Big] \\
& \quad +\prod_{k=i+1}^{N-1}\left(1+\delta L_k\left(t-\right)\right)\int_{\mathbb{R}^k}\Big[\prod_{j=i+1}^{N-1}\Big(1+\frac{\delta L_j\left(t-\right)\psi_j\left(t,X_{t-},z\right)}{1+\delta L_j\left(t\right)}\Big)-1\Big]\big(\mu-\nu^N_{X_{t-}}\big)\left(dt,dz\right).
\end{aligned}
\tag{B.4}
$$

Due to the observed martingale property of $\eta_{N,\alpha,\beta}$ under $\mathcal{Q}^N$, the drift term $(\ldots)\,dt$ may be eliminated when inserting (B.4) into Equation (B.3). Consequently, (B.3) can be rewritten as

$$
\begin{aligned}
d\eta_{N,\alpha,\beta}\left(t\right) = \delta\,\frac{B\left(0,T_N\right)}{C_{\alpha,\beta}\left(0\right)}\Bigg(&\sum_{i=\alpha}^{\beta-1}\prod_{k=i+1}^{N-1}\left(1+\delta L_k\left(t\right)\right) \\
&\times\Big[\sum_{j=i+1}^{N-1}\frac{\delta L_j\left(t-\right)}{1+\delta L_j\left(t-\right)}\sigma_j\left(t,X_{t-}\right)' dW^N\left(t\right) \\
&+\int_{\mathbb{R}^k}\Big[\prod_{j=i+1}^{N-1}\big(1+\frac{\delta L_j\left(t-\right)\psi_j\left(t,X_{t-},z\right)}{1+\delta L_j\left(t\right)}\big)-1\Big]\big(\mu-\nu^N_{X_{t-}}\big)\left(dt,dz\right)\Big]\Bigg) \\
= \delta\,\frac{B\left(0,T_N\right)}{C_{\alpha,\beta}\left(0\right)}\,\frac{1}{B\left(t,T_N\right)}\Bigg(&\sum_{i=\alpha}^{\beta-1}B\left(t,T_{i+1}\right) \\
&\times\Big[\sum_{j=i+1}^{N-1}\frac{\delta L_j\left(t-\right)}{1+\delta L_j\left(t-\right)}\sigma_j\left(t,X_{t-}\right)' dW^N\left(t\right) \\
&+\int_{\mathbb{R}^k}\Big[\prod_{j=i+1}^{N-1}\big(1+\frac{\delta L_j\left(t-\right)\psi_j\left(t,X_{t-},z\right)}{1+\delta L_j\left(t\right)}\big)-1\Big]\big(\mu-\nu^N_{X_{t-}}\big)\left(dt,dz\right)\Big]\Bigg).
\end{aligned}
$$

where (B.2) was re-substituted. Expanding the term with the annuity $C_{\alpha,\beta}\left(t\right)$ furthermore gives

$$
\begin{aligned}
\frac{d\eta_{N,\alpha,\beta}\left(t\right)}{\eta_{N,\alpha,\beta}\left(t-\right)} =& \delta\sum_{i=\alpha}^{\beta-1}\frac{B\left(t,T_{i+1}\right)}{C_{\alpha,\beta}\left(t\right)}\Big[\sum_{j=i+1}^{N-1}\frac{\delta L_j\left(t-\right)}{1+\delta L_j\left(t-\right)}\sigma_j\left(t,X_{t-}\right)' dW^N\left(t\right) \\
&+\int_{\mathbb{R}^k}\left[\prod_{j=i+1}^{N-1}\left(1+\frac{\delta L_j\left(t-\right)\psi_j\left(t,X_{t-},z\right)}{1+\delta L_j\left(t\right)}\right)-1\right]\big(\mu-\nu^N_{X_{t-}}\big)\left(dt,dz\right)\Big].
\end{aligned}
$$

The application of Girsanov's Theorem 2.3 then yields the claim.

C. Proof of Proposition 5.2

Using the defining property (3.1) of LIBOR rates, the expression for swap rates (3.3) may be rewritten as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{i=\alpha}^{\beta-1} \frac{1}{1+\delta L_i(t)}}{\delta \sum_{k=\alpha}^{\beta-1} \prod_{i=\alpha}^{k} \frac{1}{1+\delta L_i(t)}} . \tag{C.1}$$

Once again employing Itō's Lemma 2.1, we find

$$\begin{aligned} dS_{\alpha,\beta}(t) = &(\ldots)\, dt + \sum_{j=\alpha}^{\beta-1} \frac{\partial S_{\alpha,\beta}(t)}{\partial dL_j(t)} L_j(t-)\, \sigma_j(t, X_{t-})'\, dW^{j+1}(t) \\ &+ \int_{\mathbb{R}^k} \left[\frac{1 - \prod_{i=\alpha}^{\beta-1} \frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}}{\delta \sum_{k=\alpha}^{\beta-1} \prod_{i=\alpha}^{k} \frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}} - S_{\alpha,\beta}(t-) \right] \left(\mu - \nu_{X_{t-}}^{j+1}\right)(dt, dz), \end{aligned}$$

where all drift terms were summarized in $(\ldots)\, dt$. Division by $S_{\alpha,\beta}(t-)$ on both sides yields

$$\begin{aligned} \frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t-)} &= (\ldots)\, dt + \sum_{j=\alpha}^{\beta-1} \frac{\partial S_{\alpha,\beta}(t)}{\partial dL_j(t)} \frac{L_j(t-)}{S_{\alpha,\beta}(t-)} \sigma_j(t, X_{t-})'\, dW^{j+1}(t) \\ &\quad + \int_{\mathbb{R}^k} \frac{1}{S_{\alpha,\beta}(t-)} \left[\frac{1 - \prod_{i=\alpha}^{\beta-1} \frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}}{\delta \sum_{k=\alpha}^{\beta-1} \prod_{i=\alpha}^{k} \frac{1}{1+\delta L_i(t-)[1+\psi_i(t,X_{t-},z)]}} - S_{\alpha,\beta}(t-) \right] \left(\mu - \nu_{X_{t-}}^{j+1}\right)(dt, dz) \\ &= (\ldots)\, dt + \sum_{j=\alpha}^{\beta-1} x_j(t) \sigma_j(t, X_{t-})'\, dW^{j+1}(t) \qquad \text{(C.2)} \\ &\quad + \int_{\mathbb{R}^k} \psi_{\alpha,\beta}(t, X_{t-}, z) \left(\mu - \nu_{X_{t-}}^{j+1}\right)(dt, dz) \Big], \end{aligned}$$

where the drift term is appropriately adjusted, $x_j(t) := \partial S_{\alpha,\beta}(t) / \partial L_j(t) \cdot L_j(t-) / S_{\alpha,\beta}(t-)$ and $\psi_{\alpha,\beta}(t, X_{t-}, z)$ is defined as in (5.5). Observe next that, using again (B.2),

$$\frac{\partial}{\partial L_j(t)} \left(1 - \prod_{i=\alpha}^{\beta-1} \frac{1}{1+\delta L_i(t)} \right) = \frac{\delta}{1+\delta L_j(t)} \prod_{i=\alpha}^{\beta-1} \frac{1}{1+\delta L_i(t)} = \frac{\delta}{1+\delta L_j(t)} \cdot \frac{B(t, T_\beta)}{B(t, T_\alpha)}$$

and

$$\frac{\partial}{\partial L_j(t)} \Big(\delta \sum_{k=\alpha}^{\beta-1} \prod_{i=\alpha}^{k} \frac{1}{1+\delta L_i(t)} \Big) = \frac{-\delta^2}{1+\delta L_j(t)} \sum_{k=j}^{\beta} \prod_{i=\alpha}^{k} \frac{1}{1+\delta L_i(t)} = \frac{-\delta^2}{1+\delta L_j(t)} \sum_{k=j}^{\beta-1} \frac{B(t, T_{k+1})}{B(t, T_\alpha)}.$$

By (3.3), the swap rate can also be written as $S_{\alpha,\beta}(t) = [B(t,T_\alpha) - B(t,T_\beta)]/C_{\alpha,\beta}(t)$. Hence, applying the ordinary differentiation product rule to (C.1), it follows that

$$\begin{aligned}
\frac{\partial S_{\alpha,\beta}(t)}{\partial L_j(t)} &= \frac{\partial\left[\left(1-\prod_{i=\alpha}^{\beta-1}\frac{1}{1+\delta L_i(t)}\right)\cdot\left(\delta\sum_{k=\alpha}^{\beta-1}\prod_{i=\alpha}^{k}\frac{1}{1+\delta L_i(t)}\right)^{-1}\right]}{\partial L_j(t)} \\
&= \frac{1}{C_{\alpha,\beta}(t)}\cdot\frac{\delta}{1+\delta L_j(t)}\cdot B(t,T_\beta) \\
&\qquad - \frac{B(t,T_\alpha)-B(t,T_\beta)}{C^2_{\alpha,\beta}(t)}\cdot\frac{-\delta^2}{1+\delta L_j(t)}\sum_{k=j}^{\beta-1}B(t,T_{k+1}) \\
&= \frac{1}{C_{\alpha,\beta}(t)}\cdot\frac{\delta}{1+\delta L_j(t)}\left[B(t,T_\beta)+\frac{B(t,T_\alpha)-B(t,T_\beta)}{C_{\alpha,\beta}(t)}\cdot\delta\sum_{k=j}^{\beta-1}B(t,T_{k+1})\right] \\
&= \frac{S_{\alpha,\beta}(t)\,\delta}{1+\delta L_j(t)}\left(\frac{B(t,T_\beta)}{B(t,T_\alpha)-B(t,T_\beta)}+\frac{1}{C_{\alpha,\beta}(t)}\cdot\delta\sum_{k=j}^{\beta-1}B(t,T_{k+1})\right).
\end{aligned}$$

Consequently,

$$x_j(t) = \frac{\delta L_j(t)}{1+\delta L_j(t)}\left(\frac{B(t,T_\beta)}{B(t,T_\alpha)-B(t,T_\beta)}+\frac{1}{C_{\alpha,\beta}(t)}\cdot\delta\sum_{k=j}^{\beta-1}B(t,T_{k+1})\right).$$

Dynamics (5.4) then immediately follow from employing Theorem 5.1 that allows to change in (C.2) from the forward measure $\mathcal{Q}^{i+1}$ to the swap measure $\mathcal{Q}^{\alpha,\beta}$. The martingale property of $S_{\alpha,\beta}$ with respect to $\mathcal{Q}^{\alpha,\beta}$ is used to eliminate the drift term. This proves the claim.